Fast calibration of the Libor Market Model with Jacobi stochastic volatility model

4^{th} PARTY, Sibiu

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Challenges in Solvency II (1/3)

European regulatory framework: Solvency II (2016)

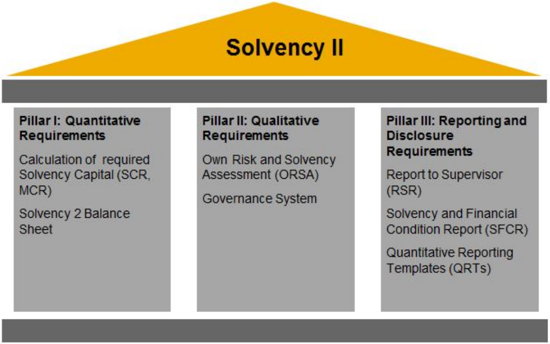

Figure 1: Solvency II regulatory framework

Challenges in Solvency II (1/3)

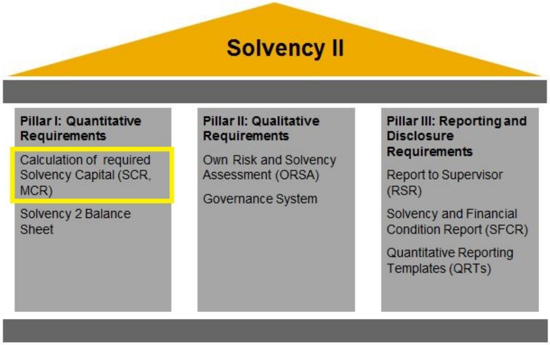

Figure 1: Solvency II regulatory framework

→ Focus on the computation of the SCR.

Challenges in Solvency II (2/3)

Insurers are exposed to **a lot of risks**: behaviour of policyholders, mortality rates, (natural) disasters, operational risks, ...

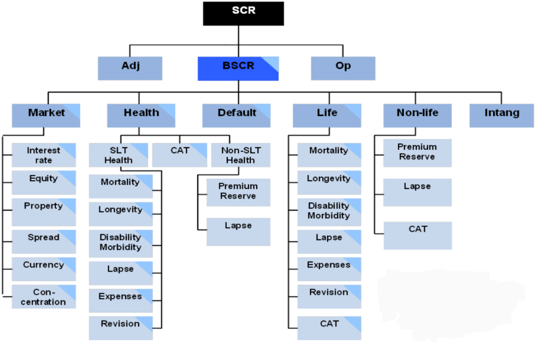

Figure 2: Structure of the standard formula

Challenges in Solvency II (2/3)

Among all these risks: the **financial risk** is chosen to be managed thanks to **mathematical financial models**.

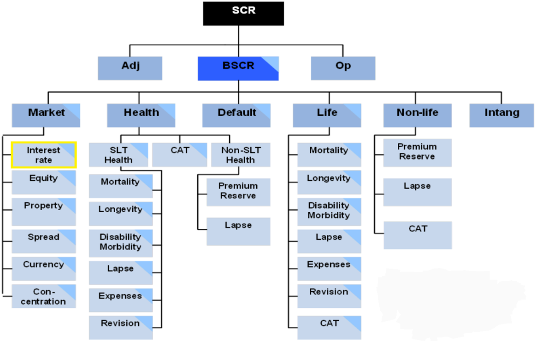

Figure 2: Structure of the standard formula

→ Major part of insurers' portfolio are *bonds* (≈80%) + other derivatives on interest-rates.

Challenges in Solvency II (3/3)

Models dedicated to interest rates are decisive, some may be complex to handle. They are asked..

- ..to be *Risk-Neutral* (No Arbitrage Assumption to obtain fair value of derivatives);
- ..to be consistent with market data (*market-consistency*): the models are calibrated to market data, they must replicate observed prices.

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$\rightsquigarrow$ Calibration is the key point of this second point!

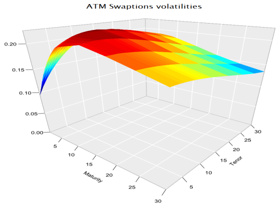

Figure 3: Market data to be replicated

Standard uses: $\approx$ 10 parameters to fit around 300 (swaption) prices.

A stochastic volatility model?

Goal: pricing of swaptions (call option on swap rate)

$$\text{Price}(\sigma^{\text{implied}}; K, T) \approx \mathbb{E}\big[\max(S_T - K, 0)\big]$$

$\rightsquigarrow$ Choice: model the financial driver S_T (swap rate) using an Ito diffusion.

The use of stochastic volatility type models is especially adapted.

The DD-SV-LMM

The considered model is the **Displaced Diffusion with Stochastic Volatility Libor Market Model** (DD-SV-LMM, here under its *normal version*, see Annex): under a convenient probability measure

$$\begin{cases} \mathrm{d}(S_t + \delta) = \sqrt{V_t}\gamma(t)\mathrm{d}W_t \\ \mathrm{d}V_t = \kappa(\theta - \xi(t)V_t)\mathrm{d}t + \epsilon\sqrt{V_t}\mathrm{d}Z_t \\ (S_0, V_0) \in \mathbb{R} \times \mathbb{R}_+^* \end{cases}$$

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[ABBD2017] proposed to take advantage of the **affine property** of the model and to **approximate the density** of S_T $\Rightarrow$ computation of approximated prices

$$\text{Price} = \int_{\mathbb{R}} (s-K)_+ f_T(s)\mathrm{d}s \approx \int_{\mathbb{R}} (s-K)_+ f_T^{(N)}(s)\mathrm{d}s =: \text{Price(N)}$$

Gram-Charlier expansion

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Theorem (Cramèr [C1926])

If f_T is of finite variation in every finite interval and is such that

$$\int_{\mathbb{R}} |f_T(s)| e^{s^2/4} \mathrm{d}s < \infty$$

then $f_T^{(N)}(x) = g(x) \sum_{n=0}^{N} c_n H_n(x) \xrightarrow[N \to +\infty]{} f_T(x)$ at every continuity point x of f_T.

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- H_n polynomial function (n-th degree), explicitly known (**Hermite polynomials**)
- The coefficients c_n are **linear combination of moments** of the unknown density f_T: need a way to compute the needed moments

Gram-Charlier and stochastic volatility? (1/2)

$$X := \sqrt{V} \times G$$

with $G \sim \mathcal{N}(0, \sigma^2)$ and $V \sim \chi^2(d)$, G and V being **independent**.

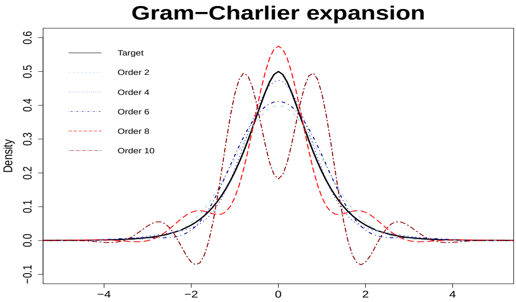

Figure 4: Gram-Charlier expansion of the density of X up to order 10 - $\sigma^2 = 0.25$ & $d = 4$

Gram-Charlier and stochastic volatility? (2/2)

$$X^{(M)} := \sqrt{\min(V, M)} \times G$$

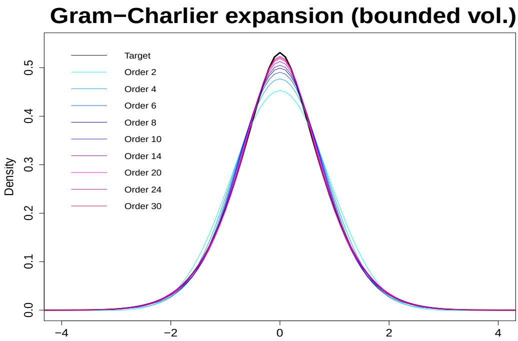

Figure 5: Gram-Charlier expansion of the density of $X^{(M)}$ up to order 30 - $M = 4$

Condition: $\sigma^2 M < 2$

Skinny distribution tail

Unbounded volatility process: Gram-Charlier approximation unlikely to converge

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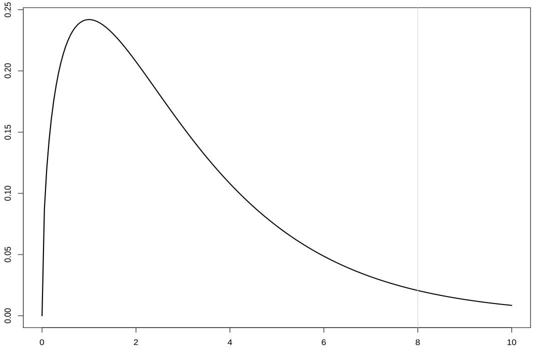
Unbounded volatility process: Gram-Charlier approximation unlikely to converge

Modelling assumption: $\mathbb{P}(V_t > v_{max}) \approx 0$ when v_{max} is large enough



Introduction of the Jacobi dynamic

Based on a work of D. Ackerer, D. Filipović and S. Pulido ([AFP2018]), we introduce the Jacobi dynamic:

$$\begin{cases} \mathrm{d}S_t & = \rho\gamma(t)\sqrt{Q(V_t)}\mathrm{d}Z_t + \sqrt{V_t - \rho^2 Q(V_t)}\gamma(t)\mathrm{d}Z_t^{\perp} \\ \mathrm{d}V_t & = \kappa\left(\theta - \xi(t)V_t\right)\mathrm{d}t + \epsilon\sqrt{Q(V_t)}\mathrm{d}Z_t \\ (S_0, V_0) & \in \mathbb{R}\times]0, v_{max}] \end{cases}$$

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- if $v_{max} T \max_{t \leqslant T} \gamma^2(t) < 2$, a Gram-Charlier expansion is theoretically allowed for the density f_T of S_T
- **Polynomial model**: we can compute any moments of S_T

What is the error? (1/2)

Theorem

There exists finite constants C, K such that

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}\left[|V_t^{\textit{Bounded}} - V_t| \right] \leqslant C / \log\left(v_{max}\right),$$

and

$$\mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} |V_t^{\textit{Bounded}} - V_t| \right] \leqslant K / \sqrt{\log\left(v_{max}\right)}.$$

What is the error? (2/2)

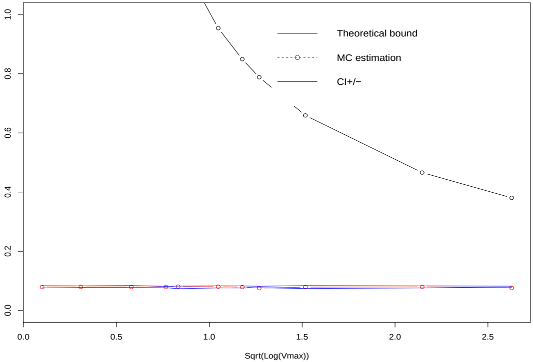

Figure 7: $\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left|V_t^{\text{Bounded}} - V_t\right|\right]$ error and theoretical bound

References

[ABBD2017] P.-E. Arrouy, and P. Bonnefoy, A. Boumezoued, L. Devineau, *Fast calibration of the Libor Market Model with Stochastic Volatility and Displaced Diffusion* (2017).

[AFP2018] D. Ackerer, D. Filipović and S. Pulido, *The Jacobi stochastic volatility model* (2018).

[C1926] H. Cramèr, *On some classes of series used in mathematical statistics* (1926).

Thank You!

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Log-normal version of the DD-SV-LMM

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In this framework, the *shifted swap rate* process $(S_t + \delta)_{t \geqslant 0}$ remains positive.