Interest rate modelling in insurance: Jacobi stochastic volatility in the Libor Market Model

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Introduction

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- Why using such a model coming from bank industry in insurance ?
- Solvency II (Solvabilité 2) is a European legislation that entered into application January 1^{st}, 2016.
- Insurers are asked to establish their risk profile in detail, in order to value the risks they are exposed to and deduce the *SCR* (Solvency Capital Requirement).
- The *SCR* theoretically guarantees that the insurer will be solvent in 1 year in 99.5% of possible market movements $\rightsquigarrow$ it is a quantile.

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- Legislation still subject to debate (regular udpdates).

Content

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Regulatory framework (1/2)

- Solvency II is composed of 3 pillars:

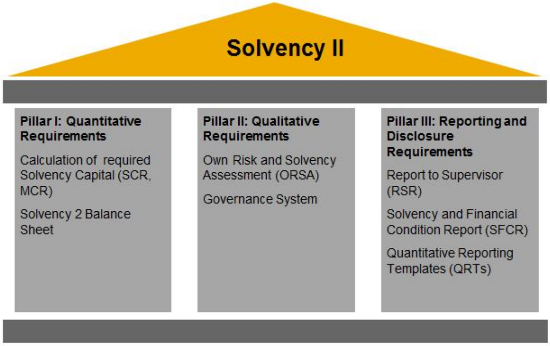

Figure: Solvency II regulatory framework

Regulatory framework (1/2)

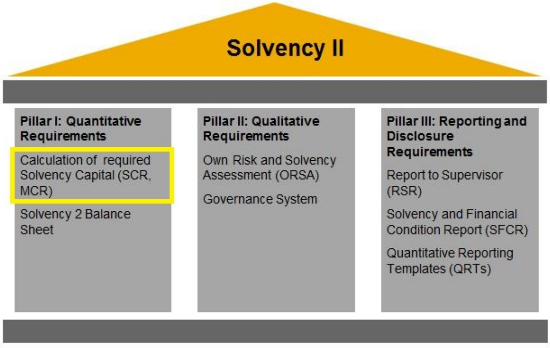

Figure: Solvency II regulatory framework

→ Focus on the SCR computation.

Regulatory framework: Solvency II (2/2)

- The *SCR* is a valuation of the risks face by insurers: behavior of policyholders (lapses), natural disaster, rise/fall of mortality, financial risks, ...

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- It can be computed either by standard formula (formula given by the regulator) or by internal model (intended to important firms).

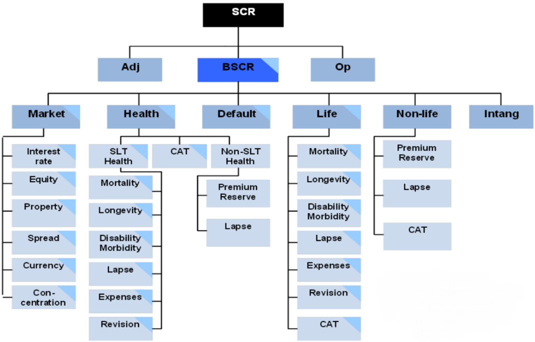

Figure: Standard formula structure

Regulatory framework: Solvency II (2/2)

- Mathematical financial models have been selected to value financial risks.
- They are incorporated in *ESG* (Economic Scenario Generators).
- The LMM+ is used to compute the *SCR* dedicated to interest-rates risk.

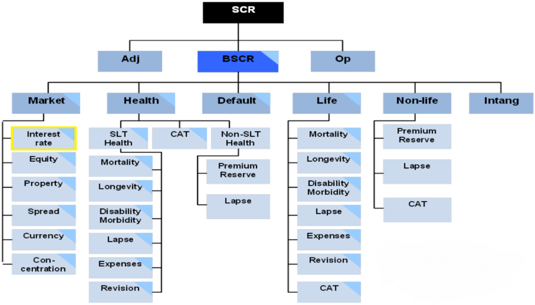

Figure: Structure of the standard formula

Market consistency

- Major part of insurers' portfolio are composed of *bonds* ($\approx$ 80%) and other derivatives on *interest-rates*.
- **Models dedicated to their modelling are decisive**, some may be complex to handle. They are asked..
  - ..to be *Risk-Neutral*;
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  - ..to be *Risk-Neutral*;
  - ..to be *market consistency*: models have to replicate market prices.
- Idea: market valuation of insurers contracts. E.g.: savings contract with guaranteed minimum rate and profits sharing. At any date t, the policyholder gets the maximum between GMR and a share associated to profits sharing reduced by managements fees:

$$e^{r_s(t)} - 1 = \max\left(e^{r_g} - 1, \mathrm{PB} \times (e^{r_{perf}(t)} - 1) - (e^{r_{fees}} - 1)\right)$$

$$\Rightarrow e^{r_s(t)} = e^{r_g} + \mathrm{PB}\left[e^{r_{perf}(t)} - \frac{\mathrm{PB} + e^{r_{fees}} + e^{r_g} - 2}{\mathrm{PB}}\right]_+$$

We show that the Fair Value of this contract at time t writes as (up to a discount factor):

$$V_t = V_0\left(e^{r_g t} + \mathrm{PB}\prod_{i=1}^{t}\mathrm{Call}\left(e^{r_{perf}(i)}, i-1, i, \frac{\mathrm{PB} + e^{r_{fees}} + e^{r_g} - 2}{\mathrm{PB}}\right)\right).$$

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Zero-Coupon bond

The LIBOR Market Model focuses on the modelling of observable quantities (following the work of Brace et al. (1997) and Jamshidian (1997); see Brigo and Mercurio (2007) for an overview of interet-rates modelling). Let $T > 0$ be a finite time horizon, and let us assume:

- the market information is generated by a N-dimensional Brownian motion $(W_t)_{t \leqslant T}$;
- there exists a Risk-Neutral probability $\mathbb{P}^*$ (equivalent to the historical one) under which discounted bond prices are martingales.

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$$\rightsquigarrow \text{Under } \mathbb{P}^*, \ \frac{dP(t,T)}{P(t,T)} = r_t dt + \sigma(t,T) \cdot dW_t^*$$

with

- $(r_t)_{t \leqslant T}$ is the risk-free rate;
- $(\sigma(t,T))_{t \leqslant T}$ is the volatility structure (adapted process);
- the correlation between Zero-Coupon bonds $P(t,T)$ and its volatility structure $\sigma(t,T)$ can be identified.

Forward rates

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- Consider a tenor structure $T_0 < T_1 < \cdots < T_K \leqslant T$. For $k \in [\![0, K-1]\!]$, the forward rate prevailing over the period $[T_k, T_{k+1}]$ is defined by:

$$F_k(t) := \frac{1}{\Delta_k}\left(\frac{P(t,T_k)}{P(t,T_{k+1})} - 1\right),\ t \leqslant T_k,\ \Delta_k := T_{k+1} - T_k.$$

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- Under $\mathbb{P}^*$, the dynamics of this is:

$$dF_k(t) = F_k(t)\gamma_k(t) \cdot \left(dW_t^* - \sigma(t, T_{k+1})dt\right)$$

where $\gamma_k(t) := \frac{1+\Delta_k F_k(t)}{\Delta_k F_k(t)}\left(\sigma(t,T_k) - \sigma(t,T_{k+1})\right)$.

Shifted forward rates

- "Late" market conditions have been such that these rates could be negative: hence the introduction of a *shift* coefficient $\delta \geqslant 0$. Recently, practitioners focus on the *shifted forward rates* (see Joshi and Rebonato (2003))

$$F_k(t) + \delta, \; t \leqslant T_k$$

and is such that

$$\mathbb{P}^* \left(\forall t \leqslant T_k : F_k(t) \geqslant -\delta \right) = 1.$$

- Under $\mathbb{P}^*$, the dynamics of shifted rates is *assumed* to be:

$$dF_k(t) = (F_k(t) + \delta)\gamma_k(t) \cdot \left(dW_t^* - \sigma(t, T_{k+1})dt \right).$$

Swap rate

- Swap rate: rate of the fixed leg in a swap (exchange) contract that will start at a future date.

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- Consider two dates $T_m < T_n \leqslant T$. The swap rate prevailing over the period $[T_m, T_n]$ is defined as:

$$S_t^{m,n} := \frac{P(t, T_m) - P(t, T_n)}{B^S(t)}, \; t \leqslant T_m,$$

with $B^S(t) := \sum_{j=m}^{n-1} \Delta_j P(t, T_{j+1})$.

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- For similar reasons, we are lead to model the *shifted swap rate*:

$$\left(S_t^{m,n} + \delta\right)_{t \leqslant T_m}.$$

- It can be shown that the shifted swap rate expresses as a deterministic function of the shifted forward rates involved during the time interval $[T_m, T_n]$.

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- It can be shown that the shifted swap rate expresses as a deterministic function of the shifted forward rates involved during the time interval $[T_m, T_n]$.
- Under $\mathbb{P}^S$, the probability measure associated to the *numéraire* $B^S(t)$, the dynamics of the shifted swap rate is:

$$dS_t^{m,n} = \sum_{j=m}^{n-1} \frac{\partial(S_t^{m,n} + \delta)}{\partial(F_j(t) + \delta)} (F_j(t) + \delta)\gamma_j(t) \cdot dW_t^S$$

where the quantities $\partial(S_t^{m,n} + \delta)/\partial(F_j(t) + \delta)$ can be analytically computed.

The volatility component

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$$\sigma(t,T) = v(t,T) \times \sqrt{V_t}$$

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with

- $(t,T) \longmapsto v(t,T)$ a deterministic function (bounded and piecewise continuous);
- $(V_t)_{t \leqslant T}$ a Cox-Ingersoll-Ross process. Its dynamics is usually specified under the Risk-Neutral measure and the dynamics under $\mathbb{P}^S$ is deduced thanks to Girsanov's theorem:

$$dV_t = \kappa(\theta - \xi(t)V_t)dt + \epsilon\sqrt{V_t}dZ_t^S$$

which Feller condition $2\kappa\theta \geqslant \epsilon^2$ ensures to have $\mathbb{P}^*\left(\forall t \leqslant T : V_t > 0\right) = 1$ (as long as $V_0 > 0$);

  - $t \longmapsto \xi(t)$ is a function appearing through the change of measures: it depends on the forward rates $(F_j(t))_{t \leqslant T_j, j \in \{0,\cdots,n-1\}}$.

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Calibration of the model

- The model is, under $\mathbb{P}^S$:

$$\begin{cases} d(S_t^{m,n} + \delta) = \sqrt{V_t} \sum_{j=m}^{n-1} \frac{\partial(S_t^{m,n} + \delta)}{\partial(F_j(t) + \delta)} \left(F_j(t) + \delta\right) \eta_j(t) \cdot dW_t^S \\ dV_t = \kappa(\theta - \xi(t)V_t)dt + \epsilon \sqrt{V_t} dZ_t^S \\ (S_0^{m,n} + \delta, V_0) \in \mathbb{R} \times \mathbb{R}_+^* \end{cases}$$

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- As it stands, the model is too complex to be calibrated: Monte-Carlo simulations are out of the operational scope.
- Based on the assumption of low variability of some ratios, these are *freezed* to their initial values.

Suggested dynamics

- The model becomes

$$
\begin{aligned}
dS_t^{m,n} &= \sqrt{V_t}\Big(\rho(t)\|\lambda^{m,n}(t)\|dW_t + \sqrt{1-\rho(t)^2}\lambda^{m,n}(t)\cdot dW_t^{S,*}\Big) \\
dV_t &= \kappa\big(\theta - \xi^0(t)V_t\big)dt + \epsilon\sqrt{V_t}dW_t,
\end{aligned}
\tag{1}
$$

with

$$
\rho(t) = \frac{d\left\langle S_\cdot^{m,n}, V_\cdot\right\rangle_t}{\sqrt{d\left\langle S_\cdot^{m,n}, S_\cdot^{m,n}\right\rangle_t d\left\langle V_\cdot, V_\cdot\right\rangle_t}}.
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- As affine dynamics, (1) offers the explicit knowledge of the characteristic function of $S^{m,n}$.
- Swaption prices $\mathbb{E}[(S_T^{m,n} - K)_+]$ can be computed... but quite long!
- Proposed calibration process: use a Gram-Charlier expansion (see Devineau et al. (2017)) based on moments of $S^{m,n}$.
- Question: convergence of Gram-Charlier expansion?

Gram-Charlier expansions

- General idea: the unknown density f is 'projected' onto a Gaussian distribution g.

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- If $f/g \in \mathcal{L}^2(g)$,

$$\frac{f^{(N)}}{g} \xrightarrow[N\to\infty]{\mathcal{L}^2(g)} \frac{f}{g},$$

with the approximating densities $f^{(N)}(x) := g(x) \times \sum_{i=0}^{N} c_i H_i(x)$.

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- The coefficients c_i are linear combinations of the moments of f.
- $f/g \in \mathcal{L}^2(g)$ means $\int_{\mathbb{R}} f(x)^2 e^{\frac{(x-\mu)^2}{2\sigma^2}} dx < \infty$.

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Theorem

For general unbounded stochastic volatility models, such as (1), this condition is not satisfied.

Gram-Charlier and stochastic volatility models (1/2)

$$X := \sqrt{V} \times G$$

with $G \sim \mathcal{N}(0, \sigma^2)$ and $V \sim \chi^2(d)$, G and V being independent.

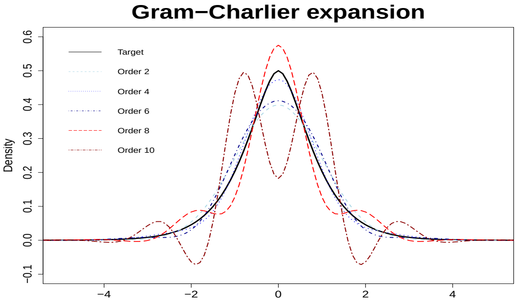

Figure: Gram-Charlier expansion of the density of X up to order 10 - $\sigma^2 = 0.25$ & $d = 4$

Gram-Charlier and stochastic volatility models (2/2)

$$X^{(M)} := \sqrt{\min(V, M)} \times G$$

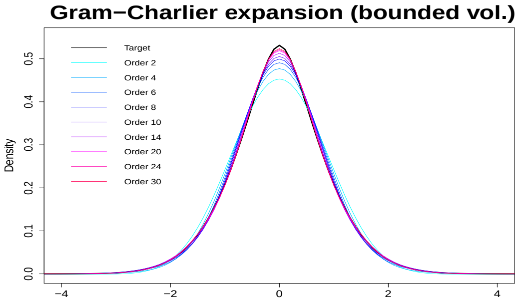

Figure: Gram-Charlier expansion of the density of $X^{(M)}$ up to order 30 - $M = 4$

Requirement: $\sigma^2 M < 2$

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Proposed dynamics (1/2)

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- We bound the volatility process to perform Gram-Charlier expansion, while preserving (we hope!) a good approximation of the swap rate distribution.
- Fix $0 \leqslant v_{min} < v_{max} \leqslant \infty$. Let us define the bounding function

$$Q(v) = \frac{(v - v_{\text{min}})(v_{\text{max}} - v)}{(\sqrt{v_{\text{max}}} - \sqrt{v_{\text{min}}})^2}$$

such that $Q(v) \in [v_{\text{min}}, v_{\text{max}}]$ for all $v \in [v_{\text{min}}, v_{\text{max}}]$. Note that $Q(v) \to v$ as $(v_{\text{min}}, v_{\text{max}}) \to (0, \infty)$.

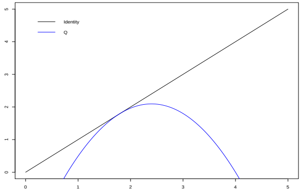

Figure: $v_{min} = 0.8$ and $v_{max} = 4$

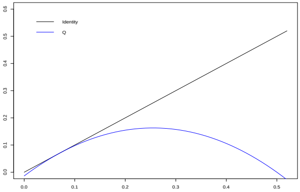

Figure: $v_{min} = 0.01$ and $v_{max} = 0.5$

Proposed dynamics (2/2)

- Based on the work Ackerer et al. (2018), we introduce the Jacobi dynamics for the stochastic volatility component:

$$\begin{aligned} dS_t^{m,n} &= \rho(t)\sqrt{Q(V_t)}\|\lambda^{m,n}(t)\| \times dW_t + \sqrt{V_t - \rho(t)^2 Q(V_t)}\lambda^{m,n}(t)\cdot dW_t^{S,*} \\ dV_t &= \kappa(\theta - \xi^0(t)V_t)dt + \epsilon\sqrt{Q(V_t)}dW_t, \end{aligned} \quad (2)$$

with

$$\sqrt{\frac{Q(V_t)}{V_t}}\rho(t) = \frac{d\left\langle S_\cdot^{m,n}, V_\cdot\right\rangle_t}{\sqrt{d\left\langle S_\cdot^{m,n}, S_\cdot^{m,n}\right\rangle_t d\left\langle V_\cdot, V_\cdot\right\rangle_t}}.$$

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$$

- If Feller condition $\frac{\epsilon^2(v_{max}-v_{min})}{(\sqrt{v_{max}}-\sqrt{v_{min}})^2} \leqslant 2\kappa \min\left(v_{max}-\theta, \theta - v_{min}\right)$ holds,

$$
\mathbb{P}^*\left(\forall t : V_t \in]v_{\text{min}}, v_{\text{max}}[\right) = 1.
$$

Proposed dynamics (2/2)

- Based on the work Ackerer et al. (2018), we introduce the Jacobi dynamics for the stochastic volatility component:

$$
\begin{aligned}
dS_t^{m,n} &= \rho(t)\sqrt{Q(V_t)}\|\lambda^{m,n}(t)\| \times dW_t + \sqrt{V_t - \rho(t)^2 Q(V_t)}\lambda^{m,n}(t)\cdot dW_t^{S,*} \\
dV_t &= \kappa(\theta - \xi^0(t)V_t)dt + \epsilon\sqrt{Q(V_t)}dW_t,
\end{aligned}
\tag{2}
$$

with

$$
\sqrt{\frac{Q(V_t)}{V_t}}\rho(t) = \frac{d\left\langle S_\cdot^{m,n}, V_\cdot\right\rangle_t}{\sqrt{d\left\langle S_\cdot^{m,n}, S_\cdot^{m,n}\right\rangle_t d\left\langle V_\cdot, V_\cdot\right\rangle_t}}.
$$

- If Feller condition $\frac{\epsilon^2(v_{max}-v_{min})}{(\sqrt{v_{max}}-\sqrt{v_{min}})^2} \leqslant 2\kappa \min\left(v_{max}-\theta, \theta - v_{min}\right)$ holds,

$$
\mathbb{P}^*\left(\forall t : V_t \in]v_{\text{min}}, v_{\text{max}}[\right) = 1.
$$

- When $(v_{min}, v_{max}) = (0, +\infty)$, we formally obtain (1).
- Weak convergence of (2) towards (1) as $(v_{min}, v_{max}) \to (0, +\infty)$.

Gram-Charlier expansion

$$\text{Assumption } (\mathbf{A}): \begin{cases} 4\kappa\theta & > \epsilon^2 \\ 2\kappa(v_{max} - \theta) & \geqslant \epsilon^2 \end{cases}$$

and

$$\text{Assumption } (\mathbf{B}): \sup_{t \in [0,T]} |\rho(t)| < 1.$$

Theorem

*Under (**A**) and (**B**), converging Gram-Charlier expansion can be performed on the unknown density of $S_T^{m,n}$ under (2) for all $v_{min} \geqslant 0$, as long as:*

$$v_{max} T \times \max_{t \leqslant T} \|\lambda^{m,n}(t)\|^2 < 2\sigma^2$$

- Application to swaptions pricing:

$$P_T(\varphi) = \int_{\mathbb{R}} f_T(s)\varphi(s)ds = \langle \varphi, \bar{f}_T \rangle_{\mathcal{L}^2(g)} = \sum_{p \geqslant 0} h_p \varphi_p$$

with
$\varphi_p = \langle \varphi, H_p \rangle_{\mathcal{L}^2(g)} = \int_{\mathbb{R}} \varphi(s) H_p(s) g(s) ds$ and $h_p = \langle H_p, \bar{f}_T \rangle_{\mathcal{L}^2(g)} = \mathbb{E}\left[H_p(S_T^{m,n})\right]$

Polynomial property

- (2) is not affine: to compute moments of $S_T^{m,n}$, we use the polynomial property of the model (see for instance Cuchiero et al. (2012) or Filipović and Larsson (2016)).
- A diffusion is said to be polynomial if its infinitesimal generator maps the set of polynomial of a given order K to itself:

$$\mathcal{A}(\mathcal{P}_K) \subset \mathcal{P}_K.$$

In this case, the action of $\mathcal{A}$ over $\mathcal{P}_K$ can be uniquely represented with a matrix A^K; then, moments can be computed using the exponential $\exp\left(A^K\right)$.

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- In our setting, the infinitesimal generator depends on time, $\mathcal{A}_t$. For $t_1 < t_2 < ... < t_J \leqslant t \leqslant T$, the action of $\mathcal{A}_t$ is represented through matrices A_j^K over each time interval. The polynomial moments can be computed as

$$\mathbb{E}\left[p(S_t^{m,n})\right] = \left(1, S_0^{m,n}, \ldots, (S_0^{m,n})^K\right) \cdot \left(\prod_{j=1}^{J} \exp\left((t_j - t_{j-1})A_{j-1}^K\right)\right) \exp\left((t - t_J)A_J^K\right) \overrightarrow{p}.$$

Convergence toward the reference dynamics (1/2)

- Weak convergence of solution of (2) towards solution of (1) as $(v_{\text{min}}, v_{\text{max}}) \to (0, \infty)$ is shown in Ackerer et al. (2018).

Convergence toward the reference dynamics (1/2)

- Weak convergence of solution of (2) towards solution of (1) as $(v_{\text{min}}, v_{\text{max}}) \to (0, \infty)$ is shown in Ackerer et al. (2018).
- We have more:

Theorem

Fix $v_{\text{min}} = 0$. There exists finite constants C_1, C_2 such that

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}\left[\left|V_t^{Jacobi} - V_t\right|\right] \leqslant C_1 / \log\left(v_{max}/V_0\right),$$

and

$$\mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} \left|V_t^{Jacobi} - V_t\right|\right] \leqslant C_2 / \sqrt{\log\left(v_{max}/V_0\right)}.$$

Convergence toward the reference dynamics (2/2)

- Previous result allows to get a pricing error: let us denote $(S_t^{m,n,J})_{t\leqslant T}$ the swap rate under dynamics (2) and $(S_t^{m,n})_{t\leqslant T}$ under dynamics (1).
- Denote the model error of pricing by

$$\epsilon_{model} = \left|\mathbb{E}^S\left[\varphi\left(S_T^{m,n}\right)\right] - \mathbb{E}^S\left[\varphi\left(S_T^{m,n,J}\right)\right]\right|.$$

Theorem

For a Lipschitz payoff φ, there exists constants K_1 and $K_2 \in \mathbb{R}$ such that

$$\epsilon_{model} \leqslant \sqrt{\frac{K_1}{\log(v_{max}/V_0)} + \frac{K_2}{v_{max}/V_0}}.$$

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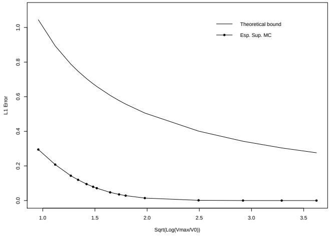
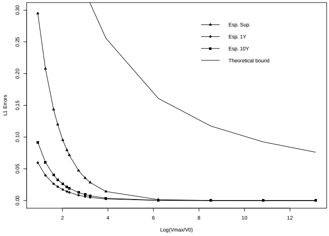
Convergence toward the reference dynamics (1/3)



$v_{min} = 0.8$ and $v_{max} = 4$



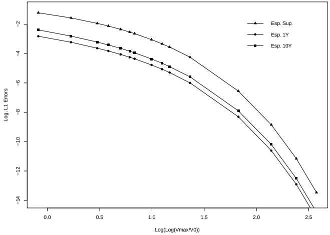
$v_{min} = 0.01$ and $v_{max} = 0.5$

Figure: $\mathbb{E}^S[\sup_{0 \leqslant s \leqslant 5} |V_s^{J,v_{max}} - V_s^C|]$, $\mathbb{E}^S[|V_1^{J,v_{max}} - V_1^C|]$ and $\mathbb{E}^S[|V_{10}^{J,v_{max}} - V_{10}^C|]$ as functions of $\sqrt{\log(v_{max}/V_0)}$ (left) and of $\log(v_{max}/V_0)$ (right).

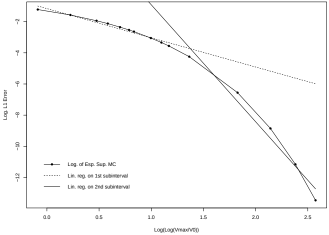
Convergence toward the reference dynamics (2/3)

Are the obtained convergence rates optimal ?

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\left[|V_t^{\text{Jacobi}} - V_t|\right] \leqslant C_1 / \log\left(v_{max}/V_0\right)$$
$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |V_t^{\text{Jacobi}} - V_t|\right] \leqslant C_2 / \sqrt{\log\left(v_{max}/V_0\right)}$$



$v_{min} = 0.8$ and $v_{max} = 4$



$v_{min} = 0.01$ and $v_{max} = 0.5$

Figure: $\log\left(\mathbb{E}^S[\sup_{0\leqslant s\leqslant 5} |V_s^{J,v_{max}} - V_s^C|]\right)$, $\log\left(\mathbb{E}^S[|V_1^{J,v_{max}} - V_1^C|]\right)$ and $\log\left(\mathbb{E}^S[|V_{10}^{J,v_{max}} - V_{10}^C|]\right)$ as functions of $\log\left(\log(v_{max}/V_0)\right)$ (left) and linear regression (right).

Convergence toward the reference dynamics (3/3)

Numerical results suggest that the convergence rate could be still improved.

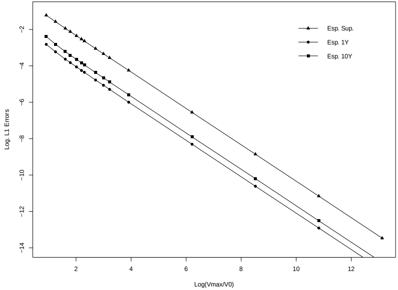


Figure: $\log\left(\mathbb{E}^S[\sup_{0\leqslant s\leqslant 5}|V_s^{J,v_{max}} - V_s^C|]\right)$, $\log\left(\mathbb{E}^S[|V_1^{J,v_{max}} - V_1^C|]\right)$ and $\log\left(\mathbb{E}^S[|V_{10}^{J,v_{max}} - V_{10}^C|]\right)$ as functions of $\log(v_{max}/V_0)$).

Gram-Charlier approximating prices (1/2)

$$\sigma^2 > \frac{v_{max} T}{2} \max_{t \leqslant T} \|\lambda^{m,n}(t)\|^2$$

is sharp to ensure the convergence of Gram-Charlier series.

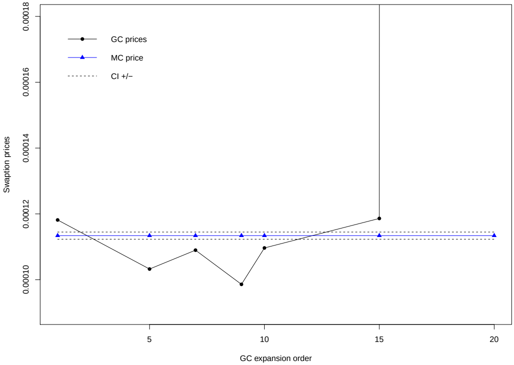
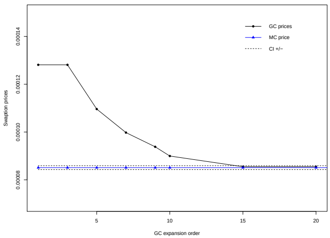


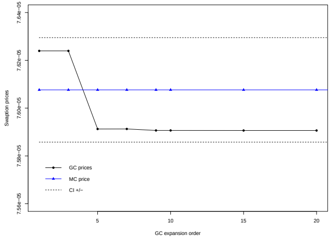
Figure: Divergence of $\sum_{p \geqslant 0}^{N} h_p \varphi_p$ as N increases.

Gram-Charlier approximating prices (2/2)

$$\text{Assumption } (\mathbf{A}): \begin{cases} 4\kappa\theta & > \epsilon^2 \\ 2\kappa(v_{max} - \theta) & \geqslant \epsilon^2 \end{cases} \quad + \quad \sigma^2 > \frac{v_{max} T}{2} \max_{t \leqslant T} \|\lambda^{m,n}(t)\|^2$$



$v_{min} = 0.8$ and $v_{max} = 4$



$v_{min} = 0.01$ and $v_{max} = 0.5$

Figure: Exemple of convergence of approximating prices to empirical ones: using a given Gaussian density as reference (left) and an adapted Gaussian distribution (matching first two moments) as reference (right).

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End of presentation

Thank you!

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