Densities approximations for stochastic volatility models in Insurance framework

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Abstract

Density approximations consist in nearing a targeted unknown probability density thanks to its *moments*. Late regulatory requirements lead insurance companies to use intensively (by performing repeated calibrations) mathematical models dedicated to specific financial drivers. Since some continuous time models (e.g. **Heston model** for index equity) can lead to time-consuming computations, performing a density approximation (the most famous one probably being the **Gram-Charlier type A** expansion) could be a solution. However, such approximations are delicate, and can only be performed (theoretically) under strict conditions. In this work, we illustrate the issues related to Gram-Charlier expansions for a toy model with stochastic volatility, and highlight the key challenges related to the extension to broader Heston-type models.

Insurance framework

Insurance and banking industries need to perform repeated calibrations of financial models. So-called *market consistent* forecasts are notably required for a variety of topics faced by insurance companies:

- projection of insurance assets and liabilities in order to value guarantees and financial options embedded in insurance contracts;
- computation of the Solvency Capital Requirement;
- implementation of intensive recalibration process;
- hedging of Variable Annuities.

Among the financial models required, those dedicated to interest rates have reached a significant complexity within the insurance market practice. Our purpose relates to the improvement of the calibration procedure of the LIBOR Market Model with Stochastic Volatility and Displaced Diffusion (DD-SV-LMM) which is now widely used.

Density approximation for stochastic volatility model

[1] proposed a new method for calibration of the DD-SV-LMM, based on approximation techniques of the density of the underlying (the swap rate, in the context of the paper). The authors approximate the (spot) price of a swaption of strike K, maturity T_m, tenor T_n, with a terminal value of the swap rate $S^n_{T_m}$, by approximating the density of the latter:

$$\pi := \mathbb{E}^{\mathbb{Q}}[(S^n_{T_m} - K)_+] = \int_{\mathbb{R}}(s-K)_+ f_m(s)\mathrm{d}s \approx \int_{\mathbb{R}}(s-K)_+ f_m^{(N)}(s)\mathrm{d}s =: \pi^{(N)}$$

As pointed out in [1], this method leads to a striking acceleration of the calibration procedure while preserving a good accuracy in prices replication.

Gram-Charlier expansion

In this section, we precise how to approximate a given target density, thanks to a Gram-Charlier (type A) expansion. We refer to [2] for more details. Let f be a probability density on $\mathbb{R}$. We aim at approximating it, thanks to a reference probability density, the Normal one:

$$g(x) := \frac{e^{-x^2/2}}{\sqrt{2\pi}}.$$

The Hermite polynomials $(H_0, H_1, \cdots, H_n)$ form an orthonormal basis of the Hilbert space $\mathcal{L}^2(g) := \{h \text{ measurable} : \int_{\mathbb{R}} |h(x)|^2 g(x)\mathrm{d}x < \infty\}$. Assuming that the likelihood-ratio f/g lies in $\mathcal{L}^2(g)$ is enough to expand it in terms of the Hermite polynomials. But, the sufficient condition is hardly met for usual distributions. [3] proved that if f is of finite variation and is such that

$$\int_{\mathbb{R}} |f(u)| e^{u^2/4}\mathrm{d}u < \infty,$$

then a Gram-Charlier expansion is possible.

Theorem (Cramér):

At every continuity point x of f:

$$\lim_{N\to+\infty} g(x)\sum_{n=0}^{N} c_n H_n(x) = f(x).$$

Note that the coefficients $(c_n)_{n\in\mathbb{N}}$ only depends on the moments of f.

Application to stochastic volatility model

We consider the following general stochastic volatility model:

$$\begin{cases} \mathrm{d}S_t &= u(V_t)\lambda(t)S_t\mathrm{d}B_t \\ \mathrm{d}V_t &= b(t,V_t)\mathrm{d}t + h(t,V_t)\mathrm{d}W_t \end{cases}, \quad (1)$$

with the correlation structure defined by $\langle B_., W_.\rangle_t = \rho t$, $t \geq 0$. We assume the function u takes its value in $\mathbb{R}_+$, is not constant and that the function λ is not equal to zero. For a $T > 0$, f_T is the density of S_T, and one can show that $\int_{\mathbb{R}} \frac{f_T^2(u)}{g(u)}\mathrm{d}u = \infty$, and thus $f/g \notin \mathcal{L}^2(g)$. Gram-Charlier series provide a fast calibration algorithm when the expansion order is carefully chosen (4 in the context of the DDSVLMM), as showed in [1]. However, the extension of this method to general orders appears challenging according to this last observation. Therefore, additional investigations are required to explore modelling frameworks that are suited for systematic Gram-Charlier expansions.

Results on a toy model

To illustrate how sensitive a Gram-Charlier type A expansion is, we take a look at two simple stochastic volatility model. First, we consider that our variable of interest is:

$$X := \sqrt{V} \times G$$

with $G \sim \mathcal{N}(0, \sigma^2)$ and $V \sim \chi^2(4)$, G and V being **independent**. The use of a chi-square distribution is motivated by the use of Cox-Ingersoll-Ross volatility dynamics in Heston-type models. The moments of X are analytically known:

$$\mathbb{E}[X^{2n}] = (n+1)(2n)!\sigma^{2n}$$

. The second model is just as elementary as the first one, since it is a slight modification of it:

$$X^{(M)} := \sqrt{V \wedge M} \times G,$$

with $M \in \mathbb{R}_+$. Again, the moments of $X^{(M)}$ are (semi-)analytically known, allowing to perform a Gram-Charlier expansion. The observed divergence for the first model (Fig. 1) is mainly due to the non-boundedness of the volatility variable. Note that the successive approximations are even not probability densities. This is why the 'volatility variable' has been bounded in the second model.

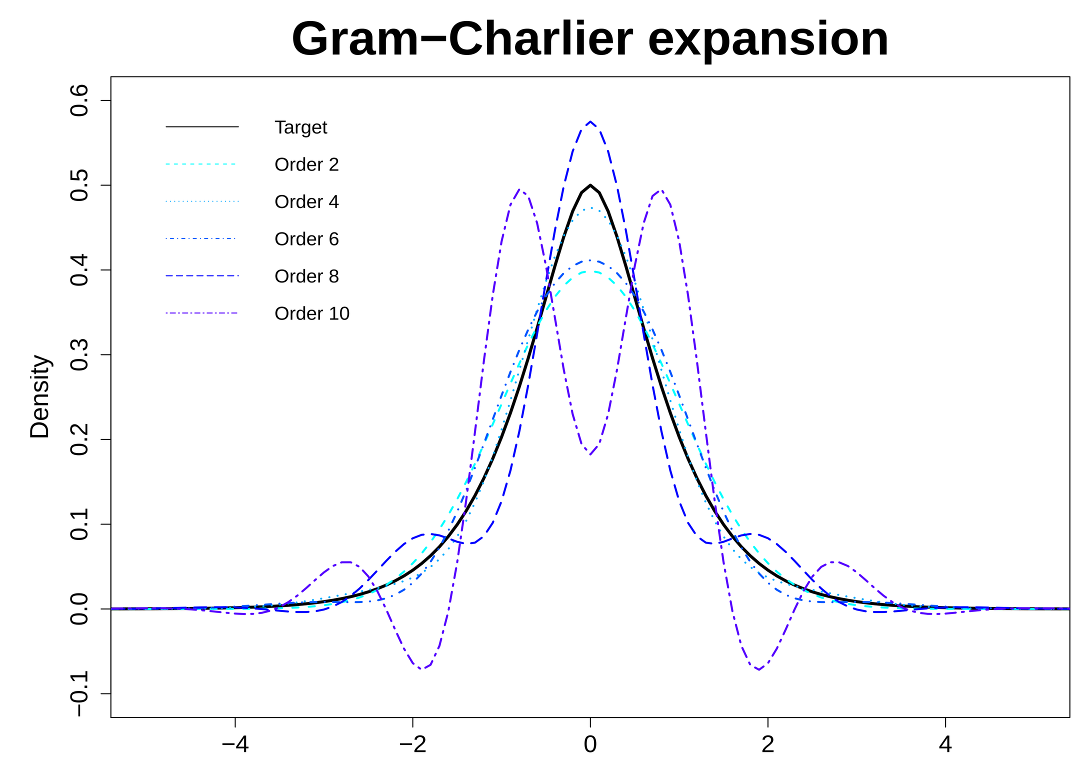

Figure 1:Gram-Charlier expansion of the density of X up to order 8 - $\sigma^2 = 0.25$

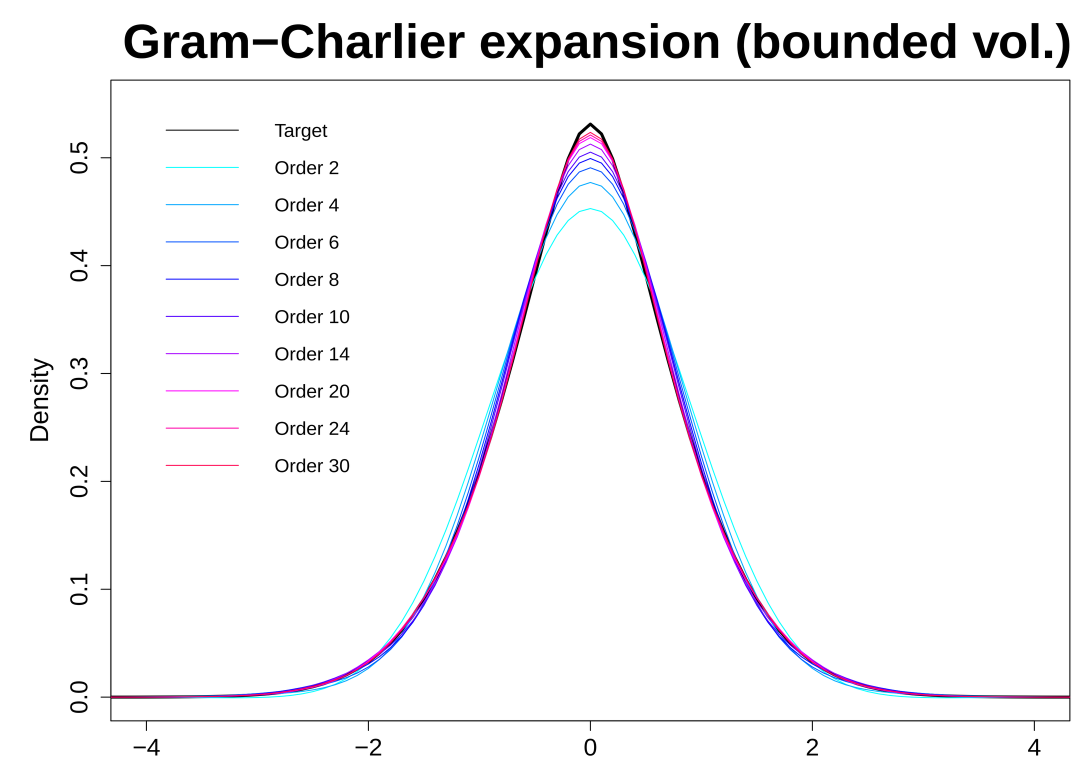

Figure 2:Gram-Charlier expansion of the density of $X^{(M)}$ up to order 30 - $M = 4$

Under the assumption (A): $\sigma^2 M < 2$, the Gram-Charlier expansion is theoretically possible. The illustrated results in Fig. 2 empirically confirm it. Note that (A) means that the variance of our variable of interest can not be arbitrary large.

References

[1] Laurent Devineau, Pierre-Edouard Arrouy, Paul Bonnefoy, and Alexandre Boumezoued. Fast calibration of the libor market model with stochastic volatility and displaced diffusion. *arXiv preprint arXiv:1706.00263*, 2017. submitted.

[2] Damir Filipović, Eberhard Mayerhofer, and Paul Schneider. Density approximations for multivariate affine jump-diffusion processes. *Journal of Econometrics*, 176(2):93–111, 2013.

[3] Harald Cramér. *On some classes of series used in mathematical statistics.* Hoffenberg, 1926.