Don’t choose between workload balance and makespan minimization

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Joint work with Sébastien Deschamps

A modeling exercise

Input.

- Set U of tasks, $u \in U$ requires d_u minutes to be completed
- Set W of workers
- Bipartite graph $G = (U \cup W, E)$: edge uw = worker w has the skill for task u
- Parallel work on a task allowed

Output. Time spent by each worker on each task so as to minimize the maximum workload

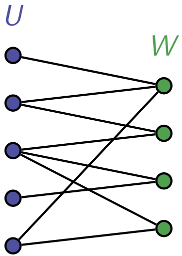
Exercise: Model this as a linear program

workload of a worker = total time worked by him

If there is no break, maximum workload = makespan
makespan = time required to complete all tasks

A modeling exercise, cont'd

$$
\begin{aligned}
\text{minimize} \quad & \max_{w \in W} \sum_{e \in \delta(w)} x_e \\
\text{s.t.} \quad & \sum_{e \in \delta(u)} x_e = d_u \quad && \forall u \in U \\
& x_e \in \mathbb{R}_+ && \forall e \in E
\end{aligned}
$$



A student: "If workload balance is optimized, then maximal workload is minimized."

A modeling exercise, cont'd

$$\begin{aligned} \text{minimize} \quad & \max_{w} \sum_{e \in \delta(w)} x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(u)} x_e = d_u \quad && \forall u \in U \\ & x_e \in \mathbb{R}_+ && \forall e \in E \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & \max_{w, w'} \sum_{e \in \delta(w)} x_e - \sum_{e \in \delta(w')} x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(u)} x_e = d_u \quad && \forall u \in U \\ & x_e \in \mathbb{R}_+ && \forall e \in E \end{aligned}$$

A student: "Any optimal solution of the program on the right is an optimal solution of the problem on the left."

Polymatroids

The student is right.

Theorem Fujishige 1980

Every polymatroid admits a unique lexicographically minimal point of its base polyhedron. Moreover, this point is also lexicographically maximal.

Base polyhedron of a polymatroid on a ground set $W =$

$$\{z \in \mathbb{R}_+^W : z(S) \leqslant f(S)\, \forall S \subseteq W,\, z(W) = f(W)\}$$

where f is submodular $\big(f(S) + f(T) \geqslant f(S \cap T) + f(S \cup T)\big)$

In our case: the map $S \subseteq W \mapsto d(N(S)) \in \mathbb{R}_+$ is submodular.

Beyond polymatroid

$$\begin{aligned} \text{minimize} \quad & \max_{w} \sum_{e \in \delta(w)} b_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(u)} a_e x_e = d_u \quad & \forall u \in U \\ & x_e \in \mathbb{R}_+ \quad & \forall e \in E \end{aligned}$$

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where the a_e and b_e are positive real numbers.

Theorem Deschamps–M. 2024+

Any optimal solution of the program on the right is an optimal solution of the problem on the left (provided that the program on the right has a nonzero optimal value).

Beyond polymatroid

$$\begin{aligned} \text{minimize} \quad & \max_{w} \sum_{e \in \delta(w)} b_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(u)} a_e x_e = d_u \quad && \forall u \in U \\ & x_e \in \mathbb{R}_+ && \forall e \in E \end{aligned}$$

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where the a_e and b_e are positive real numbers.

Interpretation:

- a_{uw} = efficiency of worker w on task u
- b_{uw} = disutility of worker w on task u
- "If workload cannot be perfectly balanced, then optimizing workload balance minimizes maximal workload."

Comments

Implies the result of the beginning (special case $a_e = b_e = 1$ for all $e \in E$)

Proof:

- elementary and short (less than one page)
- work by contradiction
- improve an optimal solution of one problem using an optimal solution of the other ("transfers")

Realm of linear programming (in particular, polynomial algorithms)

Extensions?

- x_e in $\mathbb{Z}_+$?

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♣ x_e in $\mathbb{Z}_+$?

result holds when $a_e = b_e = 1$ for all $e \in E$ (Frank–Murota, 2022); counterexamples otherwise.

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♣ x_e in $\mathbb{R}$?

next slides

Negative numbers allowed

$$\begin{aligned} \text{minimize} \quad & \max_{w} \sum_{e \in \delta(w)} b_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(u)} a_e x_e = d_u \quad && \forall u \in U \\ & x_e \in \mathbb{R} && \forall e \in E \end{aligned}$$

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where the a_e and b_e are positive real numbers.

Proposition Deschamps–M. 2024+

Any optimal solution of the program on the left is an optimal solution of the problem on the right.

... but the program on the left has not always an optimal solution...

Negative numbers allowed

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where the a_e and b_e are positive real numbers.

Proposition Deschamps–M. 2024+

Given any feasible solution, there is a feasible solution with an objective value on the left that is at least as good, and with an objective value equal to 0 on the right.

Comments

Concrete implication: not clear

Quite easy to establish: realm of linear algebra

Other "dependencies"

$$\begin{aligned} \text{minimize} \quad & \max_{w} f_w(x_{\delta(w)}) & \\ \text{s.t.} \quad & f_u(x_{\delta(u)}) = d_u & \forall u \in U \\ & x_e \in \mathbb{R} & \forall e \in E \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & \max_{w,w'} f_w(x_{\delta(w)}) - f_{w'}(x_{\delta(w')}) & \\ \text{s.t.} \quad & f_u(x_{\delta(u)}) = d_u & \forall u \in U \\ & x_e \in \mathbb{R} & \forall e \in E \end{aligned}$$

where the maps f_v are continuous and componentwise increasing self-bijections of $\mathbb{R}$.

Theorem Deschamps–M. 2024+

Given any feasible solution, there is a feasible solution with an objective value on the left that is at least as good, and with an objective value equal to 0 on the right.

Ingredients of the proof

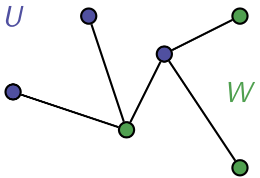
Lemma ("Implicit function theorem") Deschamps–M. 2024+

Let $f\colon \mathbb{R}^k \to \mathbb{R}$ be a componentwise increasing self-bijection of $\mathbb{R}$. Given $y \in \mathbb{R}^{k-2}$, there exists a continuous decreasing self-map φ of $\mathbb{R}$ such that $\varphi(t)$ is the unique number such that $f(y, \varphi(t), t) = 0$ for $t \in \mathbb{R}$.

- Natural and not difficult to prove
- Reminds the implicit function theorem from calculus
- Should have been already noticed (?)

Ingredients of the proof, cont'd

$$X := \{x \in \mathbb{R}^E \colon f_u(x_{\delta(u)}) = d_u,\ \forall u \in U\}$$



Lemma Deschamps–M. 2024+

Suppose that G is a tree. Then there is a unique $x^\star$ minimizing $\max_w f_w(x_{\delta(w)})$ over X. Moreover, all $f_w(x^\star_{\delta(w)})$ are equal.

- Natural but less immediate to prove
- Relies on the previous "implicit function lemma"

Comments

Proof provides an (approximation) algorithm

- assuming oracle to "invert" the f_v
- binary-search over an arbitrary spanning tree

Proof reminds of the Poincaré–Miranda theorem

Theorem Miranda (1940)

Consider continuous functions $f_1, \ldots, f_n \colon [-1,1]^n \to \mathbb{R}$ such that f_i is nonpositive when $x_i = -1$ and nonnegative when $x_i = 1$. Then there is a point where all functions are 0 simultaneously.

Thank you