Envy-free division of a cake: the poisoned case, and other variations

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Genesis, Chapter 13

From the Negev Abram went from place to place until he came to Bethel, to the place between Bethel and Ai where his tent had been earlier and where he had first built an altar. There Abram called on the name of the Lord.

Now Lot, who was moving about with Abram, also had flocks and herds and tents. But the land could not support them while they stayed together, for their possessions were so great that they were not able to stay together. And quarreling arose between Abram's herders and Lot's. The Canaanites and Perizzites were also living in the land at that time.

So Abram said to Lot, "Let's not have any quarreling between you and me, or between your herders and mine, for we are close relatives. Is not the whole land before you? Let's part company. If you go to the left, I'll go to the right; if you go to the right, I'll go to the left."

Dividing a cake

Theorem (Abraham, 1850 BC)

To divide a cake between two people in an envy-free manner, let one person cut the cake and let the other choose.

Envy-free cake cutting: the traditional setting

Envy-free cake sharing

A cake has to be shared between people.

It will be divided into as many pieces as there are people.

Each person will be assigned a piece.

Envy-free sharing of a cake: each person prefers his piece.

Envy-free cake sharing

A cake has to be shared between people.

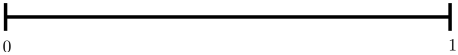
It will be divided into as many pieces as there are people.

Each person will be assigned a piece.

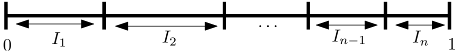
Envy-free sharing of a cake: each person is at least as happy with his piece than with any other piece.

Model

♣ n players: $i = 1, \ldots, n$

♣ Cake: $[0, 1] =$ 

♣ Division of the cake: partition of $[0, 1]$ into n pairwise disjoint intervals $I_1, \ldots, I_n$.



♣ Player i has an absolute continuous strictly positive measure μ_i on $[0, 1]$.

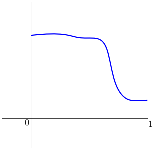
♣ Envy-free sharing: division of the cake $I_1, \ldots, I_n$ and assignment $\pi : \{\text{players}\} \longrightarrow \{\text{pieces}\}$
such that

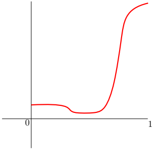
- ⋆ π is bijective.
- ⋆ $\mu_i(I_{\pi(i)}) \geqslant \mu_i(I_{\pi(j)})$ for every two players i, j.

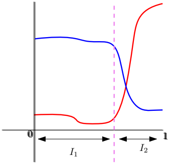
Example with 2 players

- 2 players: Alice and Bob
- Cake: $[0,1] =$ 

- Example $\mu_X(I) = \int_I f_X(u)\mathrm{d}u$

with $f_A =$ 

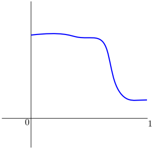
$f_B =$ 

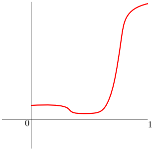
- Sharing: 

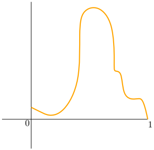
$\pi(A) = 1$, $\pi(B) = 2$

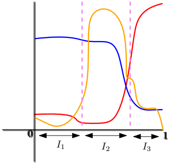
Example with 3 players

- 3 players: Alice, Bob, and Charlie
- Example $\mu_X(I) = \int_I f_X(u)\mathrm{d}u$ with

$f_A =$ 

$f_B =$ 

$f_C =$ 

- Sharing: 

$\pi(A) = 1$, $\pi(B) = 3$, $\pi(C) = 2$.

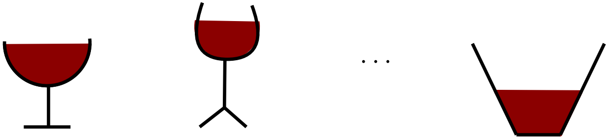
Existence of envy-free divisions

Theorem (Stromquist, Woodall, 1980)

No matter how many players there are, there is always an envy-free sharing.

Wine

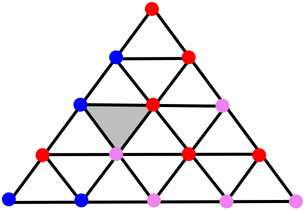
Other things than lands and cakes can be shared.



Constructive proof

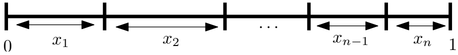
Su (1998) proposed an elegant proof based on Sperner's lemma.

$\Rightarrow$ algorithmic proof (path-following, pivot) for finding an approximate envy-free sharing.



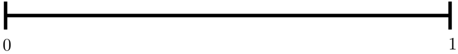
Sperner's lemma

Space of division:
$\triangle^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$



More general model

- n players: $i = 1, \ldots, n$

- Cake: $[0, 1] =$ 

- Division of the cake: partition $\mathcal{I} = (I_1, \ldots, I_n)$ of $[0, 1]$ into n pairwise disjoint intervals.

- Player i has a preference function:

$$p_i\colon \{\text{divisions}\} \to 2^{\{\text{pieces}\}} \setminus \{\varnothing\}.$$

- Given a division $\mathcal{I} = (I_1, \ldots, I_n)$:
  player i is happy with the pieces I_j such that $j \in p_i(\mathcal{I})$.

- Envy-free sharing: division $\mathcal{I}$ and assignment $\pi\colon \{\text{players}\} \longrightarrow \{\text{pieces}\}$ such that
  - $\star$ π is bijective.
  - $\star$ $\pi(i) \in p_i(\mathcal{I})$ for every player i.

A more general theorem

- Preference function p_i is closed if

$$\lim_{k\to\infty} \mathcal{I}^k = \mathcal{I} \text{ and } j \in p_i(\mathcal{I}^k)\ \forall k \quad \Longrightarrow \quad j \in p_i(\mathcal{I})$$

- Preference function p_i is hungry if

$$j \in p_i(\mathcal{I}) \Longrightarrow \lambda(I_j) \neq 0.$$

Theorem (Su, 1998)

No matter how many players there are, when all preference functions are closed and hungry, there is always an envy-free sharing.

Complexity remarks

Three theorems by Deng, Qi, and Saberi (2012).

Theorem

Finding an ε-approximation of an envy-free sharing of the cake is PPAD-complete.

Theorem

The query complexity of finding an ε-approximation is $\theta((1/\varepsilon)^{n-1})$.

Theorem

If the preference functions are Lipschitz and monotone, then there is an FPTAS for the case with $n = 3$ players.

The case $n > 3$ remains open.

A remark regarding Abraham's theorem

Abraham is able to divide the land of Canaan without consulting Lot.

He divides the land into two parts he is indifferent between.

We would like to have such a result for any number of players.

Envy-free cake cutting with a “secretive” player

Envy-free sharing with a “secretive” player

Good news!

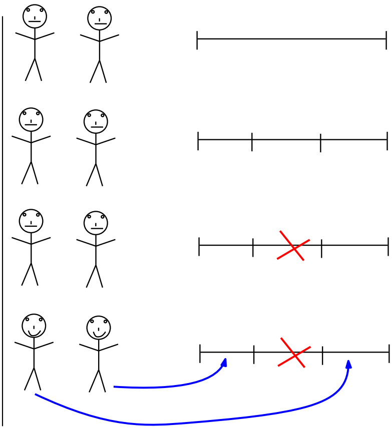
Theorem (Woodall 1980)

Consider an instance with $n-1$ players.
There exists a cake division into n pieces such that, no matter which piece is chosen, there is an envy-free assignment of the remaining pieces to the $n-1$ players.

Frick, Houston-Edwards, M. (2019), and later M., Su (2018+) proposed simpler proofs, with an algorithm (path-following, pivot).

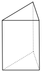
Envy-free sharing with a “secretive” player: case $n = 3$

Consider an instance with 2 players. There exists a cake division into 3 pieces such that, no matter which piece is chosen, there is an envy-free assignment of the remaining pieces to the 2 players.



Proof ingredient for $n = 3$

- Two players: Alice and Bob.

- Replace $\triangle^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ : \sum_{i=1}^3 x_i = 1\}$ by prism=  $= \triangle^1 \times \triangle^2$.

- Generalize Sperner's lemma for other polytopes than the simplex.

Envy-free cake cutting: new results

Playing with the proof

Prism$= \triangle^1 \times \triangle^2$.

Theorem

Consider an instance with 2 *players. There exists a cake division into* 3 *pieces such that, no matter which piece is chosen, there is an envy-free assignment of the remaining pieces to the* 2 *players.*

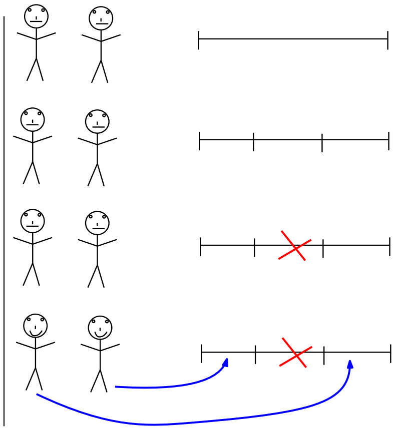
Interchanging the simplices: prism$= \triangle^2 \times \triangle^1$.

Theorem

Consider an instance with 3 *players. There exists a cake division into* 2 *pieces such that, no matter which players leaves, there is an envy-free assignment of the* 2 *pieces to the remaining players.*

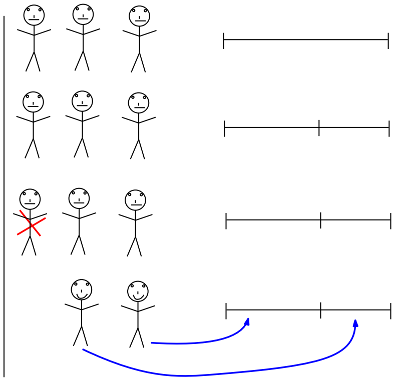
Envy-free division with a “secretive” player: case $n = 3$

Consider an instance with 2 players. There exists a cake division into 3 pieces such that, no matter which piece is chosen, there is an envy-free assignment of the remaining pieces to the 2 players.



Dual version

Consider an instance with 3 players. There exists a cake division into 2 pieces such that, no matter which players leaves, there is an envy-free assignment of the 2 pieces to the remaining players.



Playing with the proof for any n

In the proof of the "secretive" result for any n, prism $\triangle^{n-2} \times \triangle^{n-1}$.

Interchanging the left and right-hand sides: $\triangle^{n-1} \times \triangle^{n-2}$.

Theorem (M., Su 2019+)

For any instance with n players, there exists a cake division into $n-1$ connected pieces so that no matter which player leaves, there is an envy-free assignment of the pieces to the remaining $n-1$ players.

Open question: any nice interpretation?

Playing with the proof, cont'd

Theorem

Consider an instance with n players. The following holds for all integers $1 \leqslant p, q \leqslant n$.

(1) *For any subset of p players, there is a division of the cake into n pieces such that, no matter which $\lceil \frac{n-p}{p} \rceil$ pieces are selected by other players, there is an envy-free assignment of p of the remaining pieces to the p players.*

(2) *There is a division of the cake into q pieces such that, no matter which $\lceil \frac{n-q}{q} \rceil$ players leave, there is an envy-free assignment of the pieces to some q of the remaining players.*

Another corollary of the proof

- m workers and n factories.

- Company can choose for each factory the common wage: $\boldsymbol{x} \in \mathbb{R}^n_+$.

- Worker i has a nonnegative continuous utility function $u_i(j, \boldsymbol{x})$ = utility for him to work at factory j with a wage x_j when the other wages are given by the $x_{j'}$ for $j' \neq j$.

- Company has a target number k_j of workers assigned to each factory j and total budget $B > 0$ for the wages.

Theorem

If, for all i, we have $u_i(j, \boldsymbol{x}) = 0$ when $x_j = 0$, then it is possible to choose the wages and to assign k_j workers to each factory j so that no worker will strictly prefer to be assigned to another factory.

The poisoned case

Preference functions can model more

- Player i has a preference function:

$$p_i \colon \{\text{divisions}\} \to 2^{\{\text{pieces}\}} \setminus \{\varnothing\}.$$

- Given a division $\mathcal{I} = (I_1, \ldots, I_n)$:
  player i is happy with the pieces I_j such that $j \in p_i(\mathcal{I})$.

- Can be used to model:
  - Burnt or poisoned cake.
  - Corked wine.
  - Rent division.
  - Etc.

A more general theorem

- Preference function p_i is closed if

$$\lim_{k\to\infty} \mathcal{I}^k = \mathcal{I} \text{ and } j \in p_i(\mathcal{I}^k)\ \forall k \quad \Longrightarrow \quad j \in p_i(\mathcal{I})$$

- Full-division assumption: when the cake is divided into n pieces, no player is happy with the empty piece.

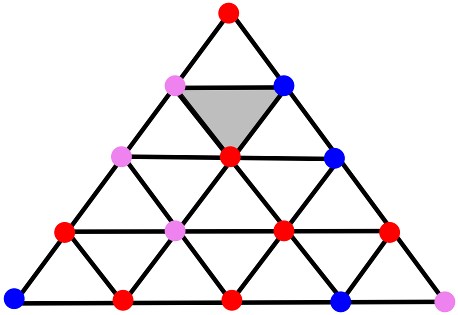
Theorem (M., Zerbib 2019)

Consider an instance with n players, with closed preference functions and the full-division assumption. If n is a prime number or is equal to 4, then there exists an envy-free division of the cake.

Conjectured by Segal-Halevi (2018) to be true for all n.

Proof ingredient

“Sperner”’s lemma with a symmetry on the boundary



Proved by Segal-Halevi (2018) for $n \leqslant 3$.

Thank you.