Combinatorial Stokes' formulae

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Abstract

For an n-dimensional pseudomanifold whose vertices get labels from a finite set, there is a "combinatorial Stokes" formula, found by Ky Fan, which links the number of simplices getting n different labels on the boundary with the number of simplices getting $n+1$ different labels. In 1998, a generalization of this formula was proved by Lee and Shih taking into account the possibility of putting several labels on each vertex. We reprove and generalize this latter combinatorial Stokes formula in a rather simple and natural way. Furthermore, some applications of the combinatorial Stokes formula of Fan are given; one of them provides a new combinatorial proof of Schrijver's theorem about Kneser graphs.

Key Words: chain map; combinatorial proof; combinatorial Stokes' formulae; Fan's combinatorial formula; Kneser graphs; labeling; multilabeling; Schrijver graphs; Sperner's lemma; Tucker's lemma; triangulation.

1 Introduction

In 1967, Ky Fan proved a combinatorial formula ([3]) in an attempt to generalize both Sperner's lemma ([11]) and Tucker's lemma ([12]). In 1974, Kuhn reproved Fan's formula in the 2-dimensional case in order to give a constructive proof of the Fundamental Theorem of Algebra ([6]). Kuhn called his formula the combinatorial Stokes' theorem because it relies the number of fully labeled simplices of the boundary of a triangulation to the number of fully labeled simplices inside the triangulation, reminding of the classical Stokes formula.

Instead of a unique label, we can put several labels on each vertex of the triangulation. Such a labeling is called a *multilabeling*, and, to our knowledge, Bapat is the first to have considered these: in 1989, he gave a multilabeling generalization of Sperner's lemma ([1]) in order to provide a constructive proof of Gale's generalization ([4]) of the KKM lemma.

In 1998, Lee and Shih found the same kind of generalization for Fan's formula ([7]).

Our aim is to give new proofs of all these results, showing that the same kind of mechanisms can always be used and to provide a simpler and more natural proof of these results. We use these techniques to prove a generalized combinatorial Stokes' formula (Theorem 6) containing Lee-Shih's result, and to show some applications of Fan's formula. One of these is a new and shorter combinatorial proof of Schrijver's theorem about Kneser graphs.

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2 Tools and notation

2.1 General notations

For a positive integer n, $[n]$ will denote the set $\{1, 2, \ldots, n\}$. For a finite set Λ, $\binom{\Lambda}{k}$ is the set of k-subsets of Λ and $\binom{\Lambda}{\leq k}$ the set of subsets of Λ whose cardinality is less or equal to k. For a sequence a_0,...,a_i,..., a_k, the sequence a_0, ..., $\hat{a}_j$, ..., a_k is the same sequence with the a_j missing.

2.2 Chain complexes

A *chain complex* $\mathcal{C}$ is a (finite or infinite) sequence of free abelian groups $(H_i)_i$ and a sequence of homomorphisms $(\partial_i : H_i \to H_{i-1})_i$ such that $\partial_{i-1} \circ \partial_i = 0$ for any i. We write:

$$\mathcal{C} : \ldots \xrightarrow{\partial_{i+1}} H_i \xrightarrow{\partial_i} H_{i-1} \xrightarrow{\partial_{i-1}} \ldots$$

The sequence (∂_i) is called the boundary operator. We often omit the index, writing ∂ instead of ∂_i.

Given two chain complexes $\mathcal{C} = (H_i, \partial_i)_i$ and $\mathcal{C}' = (H'_i, \partial'_i)_i$, a *chain map* $f : \mathcal{C} \to \mathcal{C}'$ is a sequence $(f_i)_i$ of morphisms $f_i : H_i \to H'_i$ such that $\partial'_i f_i = f_{i-1}\partial_{i-1}$.

2.3 Simplices and simplicial complexes

We give here a short introduction to the notions of simplices, simplicial complexes and chains for simplicial complexes. For a more complete study of this subject, see the book of Munkres [9].

2.3.1 Simplices and simplicial complexes

An *(abstract) simplicial complex* is a family K of subsets of a finite groundset X with the property that $\sigma' \subseteq \sigma \in \mathsf{K}$ implies $\sigma' \in \mathsf{K}$. We define the *dimension* of K: $\dim(\mathsf{K}) = \max\{|\sigma| - 1 : \sigma \in \mathsf{K}\}$. The sets in K are called *(abstract) simplices* and the dimension of a simplex σ is $\dim(\sigma) = |\sigma| - 1$. If $\dim(\sigma) = d$, we say that σ is a d-simplex.

The element of K (resp. the subsets of a simplex σ) are called *faces*. A p-face of K is a face of K of dimension p. The set of p-faces is denoted by K_p. For a p-simplex σ, the *facets* are the simplices $\sigma' \subseteq \sigma$ of dimensions $p - 1$. The *vertices* are the 0-faces of K, their set is denoted by $V(\mathsf{K})$. The *edges* are the 1-faces of K, their set is denoted by $E(\mathsf{K})$. We have thus $V(\mathsf{K}) = \mathsf{K}_0$ and $E(\mathsf{K}) = \mathsf{K}_1$.

A simplicial complex is said to be *pure* if all the maximal simplices for inclusion have the same dimension. A *k-regular* simplicial complex K is a pure simplicial complex such that any $(\dim(\mathsf{K}) - 1)$-simplex of K is contained in exactly k $\dim(\mathsf{K})$-simplices.

An n-pseudomanifold M is an n-dimensional simplicial complex whose $(n-1)$-simplices are each contained in at most two n-simplices. $\partial\mathsf{M}$, the *boundary* of the pseudomanifold M, is those $(n-1)$-simplices contained in exactly one n-simplex. A pseudomanifold having no boundary is a 2-regular simplicial complex.

Let σ be a simplex. Define two orderings of its vertex set to be equivalent if they differ from one another by an even permutation. If $\dim(\sigma) > 0$, the orderings of the vertices of σ fall into two equivalence classes. Each of these classes is called an *orientation* of σ. An *oriented simplex* is a simplex σ together with an orientation of σ. By $[v_1, \ldots, v_{p+1}]$ we denote the oriented p-simplex with vertex set $\{v_1, v_2, \ldots, v_{p+1}\}$ and the equivalence class of the particular ordering $(v_1, \ldots, v_{p+1})$.

By $-[v_1, \ldots, v_{p+1}]$, we mean the simplex $\{v_1, \ldots, v_{p+1}\}$ with the opposite orientation. Hence, we have $[v_1, \ldots, v_{p+1}] = -[v_2, v_1, v_3, v_4, \ldots, v_{p+1}]$ (for $p \geq 1$).

The *induced orientation* of $[v_1, \ldots, v_{p+1}]$ on $\{v_1, \ldots, \hat{v}_i, \ldots, v_{p+1}\}$ is $(-1)^{i+1}[v_1, \ldots, \hat{v}_i, \ldots, v_{p+1}]$. An n-pseudomanifold is said to be *oriented* if

- each n-simplex is oriented and
- for any $(n-1)$-simplex τ contained in two n-simplices, the induced orientations on τ from the two n-simplices are opposed.

Let K and L be two abstract simplicial complexes. A *simplicial map* of K into L is a mapping $f : V(\mathsf{K}) \to V(\mathsf{L})$ such that $f(\sigma) \in \mathsf{L}$ whenever $\sigma \in \mathsf{K}$.

Two abstract simplicial complexes K and L are said to be *isomorphic* if there is a simplicial map between them which is bijective and whose inverse is also simplicial. In this case, we write $\mathsf{K} \cong \mathsf{L}$, the two simplicial complexes differ only in the names of the vertices.

2.3.2 Chains for simplicial complexes

Let G be an abelian group and let K be an abstract simplicial complex. The *chain complex* $\mathcal{C}(\mathsf{K})$ is :

$$\ldots \to C_3(\mathsf{K}) \xrightarrow{\partial_{\mathsf{K}}} C_2(\mathsf{K}) \xrightarrow{\partial_{\mathsf{K}}} C_1(\mathsf{K}) \xrightarrow{\partial_{\mathsf{K}}} C_0(\mathsf{K}) \to \ldots,$$

where $C_p(\mathsf{K})$ is the free abelian group of all formal linear combinations of p-faces of K with coefficients in G. Any element c of $C_k(\mathsf{K})$ is called a *k-chain*.

We define the *boundary operator* ∂_{K} for a simplicial complex K as follows: if σ is a p-simplex $[v_1, \ldots, v_{p+1}]$, $p \geq 1$, $\partial_{\mathsf{K}}\sigma := \sum_{i=1}^{p+1}(-1)^{i+1}[v_1, \ldots, \hat{v}_i, \ldots, v_{p+1}]$. The relation $\partial_{\mathsf{K}}\partial_{\mathsf{K}} = 0$ is true for simplices since $[\ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots]$ arises twice with opposite signs. Thus $\mathcal{C}(\mathsf{K})$ is indeed a chain complex.

If f is a simplicial map of K to L, we define a collection $f_\#$ of homomorphisms $f_{\#p} : C_p(\mathsf{K}) \to C_p(\mathsf{L})$ by defining it on simplices as follows: for σ a p-simplex, we have

$$f_{\#p}(\sigma) = \begin{cases} f(\sigma) \text{ if } \dim f(\sigma) = p \\ 0 \text{ otherwise.} \end{cases}$$

$f_\#$ is then a chain map (it can be easily checked, again, first for simplices).

2.3.3 Labelings and Sperner's lemma

A *labeling* of a simplicial complex K is a map from the vertex set $V(\mathsf{K})$ into a set called the *label set*. Such a map can always be seen as a simplicial map (at least, the set of all subsets of the label set is a simplicial complex).

A Sperner labeling of a triangulation T of an n-dimensional geometric simplex σ^n is a labeling with $n+1$ labels such that (i) each vertex of σ^n is labeled with a distinct label, (ii) a vertex v of T can only get one of the labels of the vertices of its supporting face (that is, the minimal face of σ^n containing v).

The first (historically speaking) combinatorial Stokes formula is then the Sperner lemma [11]:

Theorem 1 (Sperner's lemma) *Let T be a triangulation of σ^n, an n-dimensional geometric simplex, and suppose that T gets a Sperner labeling. Then there is at least one simplex of T whose vertices are labeled with all labels.*

This is a direct consequence of many theorems of this paper.

3 Fan's formulae. Corollaries and applications

Let $\lambda : V(\mathsf{M}) \to \{-1, +1, -2, +2, \ldots, -m, +m\}$ a labeling of the vertices of an n-pseudomanifold M.

Definitions If M is oriented: For $n+1$ distinct integers $j_1, \ldots, j_{n+1}$ arranged in this order, denote by $\alpha_+(j_1, j_2, \ldots, j_{n+1})$ (resp. $\alpha_-(j_1, j_2, \ldots, j_{n+1})$) the number of those σ for which

- σ is an n-simplex of M,
- $\sigma = +[v_1, v_2, \ldots, v_{n+1}]$ (resp. $-[v_1, v_2, \ldots, v_{n+1}]$) and $\lambda(v_i) = j_i$ for $i = 1, \ldots, n+1$.

For every $(n-1)$-simplex of $\partial\mathsf{M}$, we consider the induced orientation from the unique n-simplex of M containing it.

For n distinct integers $j_1, \ldots, j_n$ arranged in this order, denote by $\beta_+(j_1, j_2, \ldots, j_n)$ (resp. $\beta_-(j_1, j_2, \ldots, j_n)$) the number of those τ such that

- τ is an $(n-1)$-simplex of $\partial\mathsf{M}$,
- $\tau = +[v_1, v_2, \ldots, v_n]$ (resp. $-[v_1, v_2, \ldots, v_n]$) and $\lambda(v_i) = j_i$ for $i = 1, \ldots, n$.

Finally, we define $\alpha := \alpha_+ - \alpha_-$ and $\beta := \beta_+ - \beta_-$.

If M is not oriented: For $n+1$ distinct integers $j_1, \ldots, j_{n+1}$, we denote by $\alpha(j_1, j_2, \ldots, j_{n+1})$ the number modulo 2 of those σ such that

- σ is an n-simplex of M,
- $\sigma = \{v_1, v_2, \ldots, v_{n+1}\}$ and $\lambda(v_i) = j_i$ for $i = 1, \ldots, n+1$.

For n distinct integers $j_1, \ldots, j_n$, we denote by $\beta(j_1, j_2, \ldots, j_n)$ the number modulo 2 of those τ such that

- τ is an $(n-1)$-simplex of $\partial\mathsf{M}$,
- $\tau = \{v_1, v_2, \ldots, v_n\}$ and $\lambda(v_i) = j_i$ for $i = 1, \ldots, n$.

Theorem 2 (Fan's formula [3]) *Let m and n be two positive integers and let M be an n-pseudomanifold. Let $\lambda : V(\mathsf{M}) \to \{-1, +1, -2, +2, \ldots, -m, +m\}$ a labeling of the vertices of M such that no edge is labeled by $-i$,$+i$ for some i (there is no antipodal edge). Then if M is oriented we have:*

$$\sum_{1 \leq j_1 < j_2 < \ldots < j_{n+1} \leq m} \alpha(-j_1, +j_2, -j_3, +j_4, \ldots, (-1)^{n+1} j_{n+1}) + (-1)^n \alpha(+j_1, -j_2, +j_3, -j_4, \ldots, (-1)^n j_{n+1}) =$$

$$\sum_{1 \leq j_1 < j_2 < \ldots < j_n \leq m} \beta(+j_1, -j_2, \ldots, (-1)^{n-1} j_n).$$

If M is not oriented, the formula holds modulo 2.

Proof: We write the proof for the oriented case. The other case is proved similarly.

λ is a simplicial map going from M into the $(m-1)$-dimensional simplicial complex C, whose faces are the subsets of $\{-1, +1, -2, +2, \ldots, -m, +m\}$ containing no pair $\{-i, +i\}$ for some $i \in [m]$ (C is the boundary of the crosspolytope).

Consider the following chain complex, which we denote by $\mathcal{C}(\mathbb{Z}^2)$:

$$\ldots \to \mathbb{Z}^2 \xrightarrow{\delta_i} \mathbb{Z}^2 \xrightarrow{\delta_{i-1}} \mathbb{Z}^2 \to \ldots,$$

where $\delta_i(x, y) = ((-1)^i x + y, x + (-1)^i y)$ (we have, indeed, $\delta_{i-1} \circ \delta_i = 0$).

We define the collection of homomorphisms $\mu : \mathcal{C}(\mathsf{C}) \to \mathcal{C}(\mathbb{Z}^2)$ as follows:

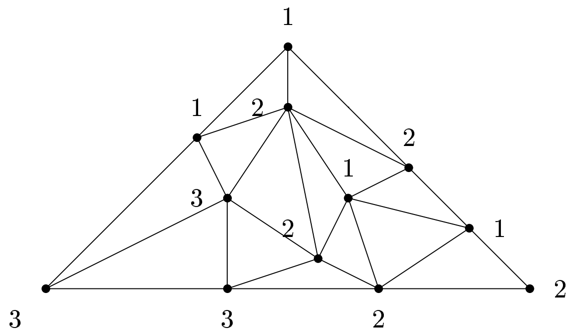

Figure 1: 2-dimensional Sperner's lemma

$$\mu_p(\sigma) = \begin{cases} (1,0) \text{ if } \sigma = [-j_1, +j_2, -j_3, +j_4, \ldots, (-1)^{p+1} j_{p+1}] \text{ with } 1 \leq j_1 < j_2 < \ldots < j_{p+1} \leq m \\ (0,1) \text{ if } \sigma = [+j_1, -j_2, +j_3, -j_4, \ldots, (-1)^{p} j_{p+1}] \text{ with } 1 \leq j_1 < j_2 < \ldots < j_{p+1} \leq m \\ 0 \text{ otherwise.} \end{cases}$$

We check now that μ is a chain map, that is: $\delta_p \mu_p = \mu_{p-1} \partial_{\mathsf{C}}$, for all $p \geq 1$. By linearity, there are only three cases to check:

1. If σ is of the form $[-j_1, +j_2, \ldots, (-1)^{p+1} j_{p+1}]$ with $1 \leq j_1 < j_2 < \ldots < j_{p+1} \leq m$:

   - $\delta_p \mu_p(\sigma) = \delta_p(1,0) = ((-1)^p, 1)$ and
   - $\mu_{p-1}(\partial_{\mathsf{C}} \sigma) = \mu_{p-1}\big(\sum_i (-1)^i [-j_1, +j_2, \ldots, \widehat{(-1)^i j_i}, \ldots, (-1)^{p+1} j_{p+1}]\big)$
     $= \mu_{p-1}([+j_2, -j_3, \ldots, (-1)^{p+1} j_{p+1}]) + \mu_{p-1}((-1)^p [-j_1, +j_2, \ldots, (-1)^p j_p])$
     $= (0,1) + ((-1)^p, 0) = ((-1)^p, 1)$.

2. If σ is of the form $[+j_1, -j_2, \ldots, (-1)^p j_{p+1}]$ with $1 \leq j_1 < j_2 < \ldots < j_{p+1} \leq m$:

   - $\delta_p \mu_p(\sigma) = \delta_p(0,1) = (1, (-1)^p)$ and
   - $\mu_{p-1}(\partial_{\mathsf{C}} \sigma) = \mu_{p-1}\big(\sum_i (-1)^i [+j_1, -j_2, \ldots, \widehat{(-1)^{i-1} j_i}, \ldots, (-1)^p j_{p+1}]\big)$
     $= \mu_{p-1}([-j_2, +j_3, \ldots, (-1)^{p+1} j_{p+1}]) + \mu_{p-1}((-1)^p [+j_1, -j_2, \ldots, (-1)^p j_p])$
     $= (1,0) + (0, (-1)^p) = (1, (-1)^p)$.

3. If $\sigma = [v_1, v_2, \ldots, v_{p+1}]$, with $|v_1| < |v_2| < \ldots < |v_{p+1}|$, and if at least a k is such that v_k and v_{k+1} have the same sign:

   - $\delta_p \mu_p(\sigma) = \delta_p 0 = 0$ and
   - $\mu_{p-1}(\partial_{\mathsf{C}} \sigma) = \mu_{p-1}\big(\sum_i (-1)^i [v_1, v_2, \ldots, \hat{v}_i, \ldots, v_{p+1}]\big)$
     $= \mu_{p-1}((-1)^{k-1} [v_1, \ldots, \hat{v}_k, \ldots, v_{p+1}]) + \mu_{p-1}((-1)^k [v_1, \ldots, \hat{v}_{k+1}, \ldots, v_{p+1}]) = 0$.

Thus $\mu\circ\lambda_{\#}$ is a chain map, and Fan's formula is a direct consequence of the commutation: $\delta_n(\mu_n \circ \lambda_{\# n}) = (\mu_{n-1} \circ \lambda_{\# n-1})\partial_{\mathsf{M}}$ applied on the formal sum of all oriented n-simplices of M. ∎

We see now some applications of this formula. The first one is Tucker's lemma [12], which is the combinatorial counterpart of the Borsuk-Ulam theorem. The second one is a purely combinatorial proof of Schrijver's theorem [10] about the chromatic number of stable Kneser graphs, shorter than Ziegler's proof of [13]. The third application is Fan's generalization of Sperner's lemma. Note that the first and the third applications are well-known (see Fan's papers [2] and [3]).

Corollary 1 (Tucker's lemma) *Let n be a positive integer and T be an antipodal triangulation of the n-sphere $\mathcal{S}^n$ (that is : if $\sigma \in \mathsf{T}$, then $-\sigma \in \mathsf{T}$, where $-\sigma$ is the symmetric of σ with respect the center of $\mathcal{S}^n$). Moreover, we ask that T refines the natural triangulation of $\mathcal{S}^n$ induced by the coordinate hyperplanes.*

Suppose that $V(\mathsf{T})$ is labeled by labels in $\{-1,+1,-2,+2,\ldots,-n,+n\}$, such that for $v \in V(\mathsf{T})$ v and $-v$ are labeled by opposite numbers (such a labeling is called an antipodal labeling). Then there exists an edge in T whose vertices are labeled by opposite numbers.

Proof: We prove in fact the following proposition:

Suppose that T is antipodally labeled with the following integers: $-1,+1,-2,+2,\ldots,-m,+m$, where m is a positive integer, in such a way that there is no edge whose vertices are labeled by opposite numbers.

Then there is an odd number of n-simplices labeled by $n+1$ integers $+j_1$, $-j_2$, ..., $(-1)^n j_{n+1}$, with $1 \leq j_1 < j_2 < \ldots < j_{n+1} \leq m$.

In particular, this implies that $m > n$, and Tucker's lemma follows.

The proof works by induction. For $n = 0$, the proof is trivial.

For $n > 0$, the proof is a direct consequence of Fan's formula in the unoriented case: let $\mathcal{S}^n_+$ be the positive hemisphere of $\mathcal{S}^n$: $x \in \mathcal{S}^n_+ \Rightarrow x_{n+1} \geq 0$. T induces a triangulation T_+ on $\mathcal{S}^n_+$. The boundary of T_+ is a triangulation of an $(n-1)$-sphere $\mathcal{S}^{n-1}$ which satisfies the conditions of the proposition above. Thus induction applies and applying Theorem 2, we know that:

$$\sum_{1\leq j_1<j_2<\ldots<j_{n+1}\leq m} \alpha(-j_1,+j_2,\ldots,(-1)^{n+1}j_{n+1}) + \alpha(+j_1,-j_2,\ldots,(-1)^n j_{n+1}) \equiv 1 \bmod 2$$

for T_+. By antipodality, this number is precisely the number of n-simplices labeled by $n+1$ integers $+j_1$, $-j_2$, ..., $(-1)^n j_{n+1}$, with $1 \leq j_1 < j_2 < \ldots < j_{n+1} \leq m$ in T. ∎

Second application: a short combinatorial proof of Schrijver's theorem. We recall the context. Let $\mathcal{H}$ be a k-regular hypergraph. The associated *Kneser graph $KG(\mathcal{H})$* is the graph whose vertices are the hyperedges of $\mathcal{H}$, and in which two vertices are connected by an edge if and only if the two corresponding hyperedges are disjoint.

For $n \geq 2k-1$, we consider the hypergraph $\mathcal{S}$ whose vertex set is $[n]$ and the hyperedges are those k-subsets A such that for $x, y \in A \subseteq [n]$, $1 < |x-y| < n-1$ (putting the elements of $[n]$ on a circle in the natural order, the hyperedges are k-subsets not having adjacent elements). The Schrijver graph of parameters n, k is then defined by $SG(n,k) := KG(\mathcal{S})$.

Denoting by $\chi(G)$ the chromatic number of a graph G, we have the following theorem, proved by Schrijver [10]:

Corollary 2 (Schrijver's theorem) $\chi(SG(n,k)) = n - 2k + 2$.

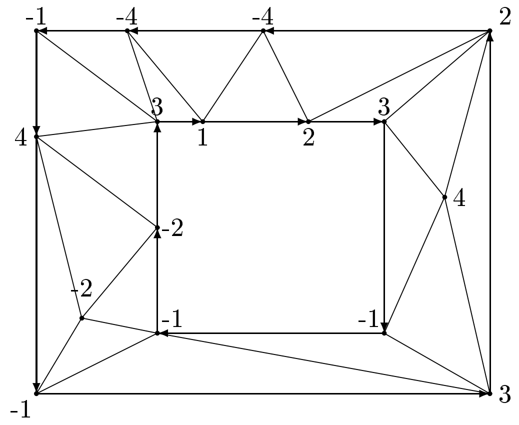

Figure 2: The triangles get the counterclockwise orientation. The arrows indicates the induced orientation on the boundary. There is only one 2-simplex of type "α" with labels: -1,3,-4; and there is only one 1-simplex of type "β" on the boundary with labels: 2,-4 in this order

We first need some definitions and notations: For a positive integer n, we write $\{+,-,0\}^n$ for the set of all *signed subsets* of $[n]$, that is, the family of all pairs (X^+, X^-) of disjoint subsets of $[n]$. Indeed, for $X \in \{+,-,0\}^n$, we can define $X^+ := \{i \in [n] : X_i = +\}$ and analogously X^-.

For sign vectors, we use the usual partial order from oriented matroid theory, which is defined componentwise with $0 \leq +$, and $0 \leq -$. Hence $X \leq Y$ if and only if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$.

By $\mathrm{alt}(X)$ we denote the length of the longest alternating subsequence of non-zero signs in X. For instance: $\mathrm{alt}(+0--+0-) = 4$, while $\mathrm{alt}(--++-+0+-) = 5$.

Finally, for a poset P, $\Delta(P)$ is the simplicial complex (the *order complex*) whose vertices are the elements of P, and whose simplices are the chains of P.

The proof is inspired by Matousek's combinatorial proof of the Lovász-Kneser theorem stating that $\chi(KG(\binom{[n]}{k})) = n-2k+2$ ([8]) and one of the main ingredients, namely the poset $\Sigma_{n,k}$, is taken from Ziegler's proof [13].

Proof: The inequality $\chi(SG(n,k)) \leq n - 2k + 2$ is easy to prove (with an explicit coloring) and well-known. So, to obtain a combinatorial proof, it is sufficient to prove the reverse inequality.

For $n \geq 2k$, we define

$$\Sigma_{n,k} := \Delta\big(\{X \in \{+,-,0\}^n : \mathrm{alt}(X) \geq 2k\big).$$

We define two more simplicial complexes, that are subcomplexes of $\Sigma_{n,k}$:

$$\Sigma^+_{n,k} := \Delta\big(\{X \in V(\Sigma_{n,k}) : X_n \in \{0,+\}\}\big)$$

and

$$\Sigma^-_{n,k} := \Delta\big(\{X \in V(\Sigma_{n,k}) : X_n \in \{0,-\}\}\big)$$

We have then the following properties:

1. $\Sigma_{n,k}^+ \cup \Sigma_{n,k}^- = \Sigma_{n,k}$.
2. $\dim(\Sigma_{n,k}^+) = \dim(\Sigma_{n,k}^-) = n - 2k$.
3. $\Sigma_{n,k}^+$ and $\Sigma_{n,k}^-$ are pseudomanifolds.
4. $\partial\Sigma_{n,k}^+ = \partial\Sigma_{n,k}^- = \Delta\big(\{X \in V(\Sigma_{n,k}) : X_n = 0\}\big) \cong \Sigma_{n-1,k}$.

All those properties are easy to prove, except maybe the third one. To prove it, we prove that $\Sigma_{n,k}$ is a pseudomanifold: take a $(n-2k-1)$-simplex of $\Sigma_{n,k}$. It has either the form (i) $X^{2k} \leq X^{2k+1} \leq \ldots \leq X^{i-1} \leq X^{i+1} \leq \ldots \leq X^n$, $2k+1 \leq i \leq n-1$, or the form (ii) $X^{2k+1} \leq \ldots \leq X^n$ or the form (iii) $X^{2k} \leq \ldots \leq X^{n-1}$, where X^i is an element of $V(\Sigma_{n,k})$ having exactly i nonzero components.

In the case (i), as we add two nonzero components to X^{i-1} to obtain X^{i+1}, the $(n-2k-1)$-simplex is exactly in two different $(n-2k)$-simplices of $\Sigma_{n,k}$.

In the case (ii), either $\mathrm{alt}(X^{2k+1}) = 2k+1$ and there are two $X < X^{2k+1}$ such that $\mathrm{alt}(X) \geq 2k$ (delete the first or the last nonzero components of X), or $\mathrm{alt}(X^{2k+1}) = 2k$ and there is exactly one j such that $X_j^{2k+1} = X_{j+1}^{2k+1} \neq 0$, implying that there are two $X < X^{2k+1}$ such that $\mathrm{alt}(X) \geq 2k$. In both cases, it means that the $(n-2k-1)$-simplex is exactly in two different $(n-2k)$-simplices of $\Sigma_{n,k}$.

In the case (iii), let j be the unique integer such that $X_j^{n-1} = 0$; there are exactly two possibilities for X_j^n: $X_j^n = -$ or $X_j^n = +$ and the $(n-2k-1)$-simplex is exactly in two different $(n-2k)$-simplices of $\Sigma_{n,k}$.

We prove now the following proposition:

Suppose that the vertices of $\Sigma_{n,k}$ are labeled with integers in the set $\{-1,+1,-2,+2,\ldots,-d,+d\}$ such that no edge is labeled $-i,+i$ for any $1 \leq i \leq d$, and such that the label of vertex X is the opposed of the label of $-X$ ($\lambda(-X) = -\lambda(X)$) for any vertex X of $\Sigma_{n,k}$. Then there is an odd number of $(n-2k)$-simplices labeled by $n-2k+1$ integers $+j_1$, $-j_2$, ..., $(-1)^{n-2k}j_{n-2k+1}$, with $j_1 < j_2 < \ldots < j_{n-2k+1}$.

The proof works by induction on n, for fixed k.

If $n = 2k$: $\Sigma_{2k,k} = \{\{(+,-,+,-,\ldots,+,-)\},\{(-,+,-,+,\ldots,-,+)\}\}$, and the proof is straightforward.

If $n > 2k$: $\Sigma_{n,k}^+$ is a pseudomanifold (property 3). Its boundary is $\Sigma_{n-1,k}$ (property 4). By induction and by Theorem 2, we have, for $\Sigma_{n,k}^+$:

$$\sum_{1 \leq j_1 < j_2 < \ldots < j_{n-2k+1} \leq d} \alpha(-j_1,+j_2,\ldots,(-1)^{n-2k+1}j_{n-2k+1}) + \alpha(+j_1,-j_2,\ldots,(-1)^{n-2k}j_{n-2k+1}) \equiv 1 \bmod 2.$$

The conclusion of the proposition follows from the fact that the number of $(n-2k)$-simplices of $\Sigma_{n,k}^-$ whose labeling is of the form $+j_1,-j_2,\ldots,(-1)^{n-2k}j_{n-2k+1}$ with $j_1 < j_2 < \ldots < j_{n-2k+1}$ is the number of $(n-2k)$-simplices of $\Sigma_{n,k}^+$ whose labeling is of the form $-j_1,+j_2,\ldots,(-1)^{n-2k+1}j_{n-2k+1}$ with $j_1 < j_2 < \ldots < j_{n-2k+1}$ (by antipodality).

The proposition is proved. We apply it to achieve the proof of Schrijver's theorem:

Let c be a coloring of $SG(n,k)$ with t colors: $c : V(SG(n,k)) \to [t]$. Our aim is to prove that $t \geq n-2k+2$.

Let $\lambda : \{+,-,0\}^n \to \{-1,+1,-2,+2,\ldots,-(t-1),+(t-1)\}$ be a labeling of the vertices of $\Sigma_{n,k}$ defined as follows: $\lambda(X^+,X^-)$ is $\pm c(S)$, where S is the stable k-subset of $[n]$, contained in X^+ or in X^-, having the smallest color. The sign indicates which of X^+ or X^- we took S from. As we require that $\mathrm{alt}(X) \geq 2k$, both X^+ and X^- contain a stable k-subset and thus $|\lambda(X^+,X^-)| \leq t-1$. By definition of a coloring, there is no antipodal edge.

We can thus apply the proposition just proved, for $d := t-1$: there is at least one $(n-2k)$-simplex of $\Sigma_{n,k}$ labeled with $1 \leq j_1 < j_2 < \ldots < j_{n-2k+1} \leq t-1$. Hence, $n-2k+1 \leq t-1$, or equivalently $n-2k+2 \leq t$. ∎

Corollary 3 (Fan's generalization of Sperner's lemma) *Let n be a positive integer and let M be an n-pseudomanifold. Let $\lambda : V(\mathsf{M}) \to [n+1]$ a labeling of the vertices of M.*

We have the the following relations:

$$\alpha(1,2,\ldots,n+1) = (-1)^n \beta(1,2\ldots,n)$$

if M is oriented, and

$$\alpha(1,2,\ldots,n+1) \equiv \beta(1,2\ldots,n) \bmod 2$$

if not.

Proof: Apply Fan's formula for the set of labels $\{+1,-2,+3,\ldots,(-1)^n(n+1)\}$. ∎

The classical Sperner lemma is a direct consequence of these relations (actually, we get much more, namely that the classical Sperner lemma holds with $\alpha = 1$).

4 Multilabelings

We consider now generalizations of the following kind: instead of one label for each vertex of a pseudomanifold, we suppose that there are many labels on each vertex.

4.1 Bapat's theorem, Lee-Shih's theorem

Definitions Let n and q be two positive integers. Let M be an n-pseudomanifold, and let $\lambda : v \in V(\mathsf{M}) \mapsto (\lambda_1(v), \ldots, \lambda_q(v))$ such that each λ_i, $i = 1, \ldots, q$, is a labeling of M. Such a λ is called a *q-multiple labeling.*

If M is oriented: For $n+1$ distinct integers $j_1, \ldots, j_{n+1}$ arranged in this order, denote by $\alpha_+(j_1, j_2, \ldots, j_{n+1})$ (resp. $\alpha_-(j_1, j_2, \ldots, j_{n+1})$) the number of those pairs (σ, f) such that

- σ is an n-simplex of M and f is an injection from $V(\sigma)$ into $[q]$,
- $\sigma = +[v_1, v_2, \ldots, v_{n+1}]$ (resp. $-[v_1, v_2, \ldots, v_{n+1}]$) and $\lambda_{f(v_i)}(v_i) = j_i$ for $i = 1, \ldots, n+1$.

For every $(n-1)$-simplex of $\partial\mathsf{M}$, we consider the induced orientation from the unique n-simplex of M containing it.

For n distinct integers $j_1, \ldots, j_n$ arranged in this order, denote by $\beta_+(j_1, j_2, \ldots, j_n)$ (resp. $\beta_-(j_1, j_2, \ldots, j_n)$) the number of those pairs (τ, g) such that

- τ is an $(n-1)$-simplex of $\partial\mathsf{M}$ and g is an injection from $V(\tau)$ into $[q]$,
- $\tau = +[v_1, v_2, \ldots, v_n]$ (resp. $-[v_1, v_2, \ldots, v_n]$) and $\lambda_{g(v_i)}(v_i) = j_i$ for $i = 1, \ldots, n$.

Finally, we define $\alpha := \alpha_+ - \alpha_-$ and $\beta := \beta_+ - \beta_-$. If M is not oriented, we define α and β as in the previous section. Now, we are in position to give Bapat's theorem:

Theorem 3 (Bapat's theorem) *Let* T *a triangulation of the* n*-simplex* σ^n *and let* $\lambda = (\lambda_i)_{i=1,\ldots,n+1}$ *be an* $(n+1)$*-multiple labeling, where the* $\lambda_i, i = 1, 2, \ldots, n+1$ *are* $n+1$ *Sperner labelings of the vertices of* T*. Then*

$$\alpha(1, 2, \ldots, n+1) = (n+1)!.$$

When all λ_i are equal, Sperner's lemma appears as a special case.

Bapat's theorem is a direct implication of the following multilabeled version of Theorem 2: as we said in the introduction, such a theorem was established by Lee and Shih in 1998 ([7]):

Theorem 4 (Lee-Shih's theorem) *Let* m, n *and* q *be three positive integers and let* M *be an* n*-pseudomanifold. Let* $\lambda = (\lambda_i)_{i=1,\ldots,q}$ *be a* q*-multiple labeling, where for each* i $\lambda_i : V(\mathsf{M}) \to \{-1, +1, -2, +2, \ldots, -m, +m\}$ *is a labeling of the vertices of* M *such that there are no* $j_1 \neq j_2$ *and* $(v_1, v_2) \in E(\mathsf{M})$ *such that* $\lambda_{j_1}(v_1) = -\lambda_{j_2}(v_2)$*.*

Then we have if M *is oriented:*

$$\sum_{1 \leq j_1 < j_2 < \ldots < j_{n+1} \leq m} \alpha(-j_1, +j_2, \ldots, (-1)^{n+1} j_{n+1}) + (-1)^n \alpha(+j_1, -j_2, \ldots, (-1)^n j_{n+1}) =$$

$$(q - n) \sum_{1 \leq j_1 < j_2 < \ldots < j_n \leq m} \beta(+j_1, -j_2, \ldots, (-1)^{n-1} j_n).$$

If M *is not oriented, the formula holds modulo 2.*

This theorem is a corollary of a more general theorem which will be stated and proved in the next section.

We obtain the formula of Theorem 2 when all λ_i are equal.

4.2 K-labelings

Now, we give a more general version of the definition of multilabelings, in order to obtain a generalization (and a simple proof) of Lee-Shih's theorem.

Definitions Let n and k be two positive integers. Let M be an n-pseudomanifold, K be a k-regular simplicial complex and let $\lambda : v \in V(\mathsf{M}) \mapsto (\lambda_w(v))_{w \in V(\mathsf{K})}$ such that each λ_w, $w \in V(\mathsf{K})$, is a labeling of M. Such a λ is called a K*-labeling*.

If M is oriented: For $n+1$ distinct integers $j_1, \ldots, j_{n+1}$ arranged in this order, denote by $\alpha_+(j_1, j_2, \ldots, j_{n+1})$ (resp. $\alpha_-(j_1, j_2, \ldots, j_{n+1})$) the number of those pairs (σ, f) such that

- σ is an n-simplex of M and $f(\sigma)$ is an n-simplex of K,
- $\sigma = +[v_1, v_2, \ldots, v_{n+1}]$ (resp. $-[v_1, v_2, \ldots, v_{n+1}]$) and $\lambda_{f(v_i)}(v_i) = j_i$ for $i = 1, \ldots, n+1$.

For every $(n-1)$-simplex of $\partial\mathsf{M}$, we consider the induced orientation from the unique n-simplex of M containing it.

For n distinct integers $j_1, \ldots, j_n$ arranged in this order, denote by $\beta_+(j_1, j_2, \ldots, j_n)$ (resp. $\beta_-(j_1, j_2, \ldots, j_n)$) the number of those pairs (τ, g) such that

- τ is an $(n-1)$-simplex of $\partial\mathsf{M}$ and $g(\sigma)$ is an $(n-1)$-simplex of K,
- $\tau = +[v_1, v_2, \ldots, v_n]$ (resp. $-[v_1, v_2, \ldots, v_n]$) and $\lambda_{g(v_i)}(v_i) = j_i$ for $i = 1, \ldots, n$.

Finally, we define $\alpha := \alpha_+ - \alpha_-$ and $\beta := \beta_+ - \beta_-$. If M is not oriented, we define α and β as in the previous section.

Theorem 5 *Let m, n and k be three positive integers, let M be an n-pseudomanifold and let K be a k-regular simplicial complex. Let $\lambda = (\lambda_w)_{w \in V(\mathsf{K})}$ be a K-labeling, where, for each $w \in \mathsf{K}$, $\lambda_w : V(\mathsf{M}) \to \{-1, +1, -2, +2, \ldots, -m, +m\}$ is a labeling of the vertices of M such that there are no $(w_1, w_2) \in E(\mathsf{K})$ and $(v_1, v_2) \in E(\mathsf{M})$ such that $\lambda_{w_1}(v_1) = -\lambda_{w_2}(v_2)$.*

Then we have if M is oriented:

$$\sum_{1 \le j_1 < j_2 < \ldots < j_{n+1} \le m} \alpha(-j_1, +j_2, \ldots, (-1)^{n+1} j_{n+1}) + (-1)^n \alpha(+j_1, -j_2, \ldots, (-1)^n j_{n+1}) =$$

$$k \sum_{1 \le j_1 < j_2 < \ldots < j_n \le m} \beta(+j_1, -j_2, \ldots, (-1)^{n-1} j_n).$$

If M is not oriented, the formula holds modulo 2.

Proof: We write the proof for the oriented case. The other case is proved similarly.

Let L be the simplicial complex whose vertex set is defined by $V(\mathsf{L}) := V(\mathsf{M}) \times V(\mathsf{K})$ and whose simplices are those subsets $\sigma = \{(v_1, w_1), \ldots, (v_{p+1}, w_{p+1})\}$ of $V(\mathsf{L})$ such that $\{v_1, \ldots, v_{p+1}\}$ is a p-simplex of M and $\{w_1, \ldots, w_{p+1}\}$ is a p-simplex of K. We give σ the same orientation that $[v_1, \ldots, v_{p+1}]$'s one.

λ induces a labeling ϕ of L: $\phi((v, w)) = \lambda_w(v)$. ϕ is a simplicial map going from L into the $(m-1)$-dimensional simplicial complex C, whose faces are the subsets of $\{-1, +1, -2, +2, \ldots, -m, +m\}$ containing no pair $\{-i, +i\}$ for some $i \in [m]$ (the boundary of the crosspolytope).

Using the same notations as in the proof of Theorem 2, we have the following equality: $\delta(\mu \circ \phi_\#) = (\mu \circ \phi_\#) \partial_{\mathsf{L}}$. Applying this formula to the formal sum of all oriented n-simplices of L, $L := \sum_{\sigma \in \mathsf{L}_n} \sigma$, and noticing that

$$\partial_{\mathsf{L}} L = k \sum_{[v_1, \ldots, v_n] \in \partial \mathsf{M}, \{w_1, \ldots, w_n\} \in \mathsf{K}_{n-1}} [(v_1, w_1), \ldots, (v_n, w_n)],$$

the formula follows. ∎

Lee-Shih's theorem is the special case when $\mathsf{K} = \binom{[q]}{\le n+1}$.

4.3 Another formula

Suppose now that we wonder about the number $\alpha_+(j_1, j_2, \ldots, j_{n+1})$ (resp. $\alpha_-(j_1, j_2, \ldots, j_{n+1})$) of those pairs (σ, f) such that

- σ is an n-simplex of M and f is **any** map from $V(\sigma)$ in $[q]$,
- $\sigma = +[v_1, v_2, \ldots, v_{n+1}]$ (resp. $-[v_1, v_2, \ldots, v_{n+1}]$) and $\lambda_{f(v_i)}(v_i) = j_i$ for $i = 1, \ldots, n+1$.

using the number $\beta_+(j_1, j_2, \ldots, j_n)$ (resp. $\beta_-(j_1, j_2, \ldots, j_n)$) of pairs (τ, g) such that

- τ is an $(n-1)$-simplex of M and g is **any** map from $V(\tau)$ in $[q]$, $\lambda_{g(v_i)}(v_i) = j_i$ for $i = 1, \ldots, n$.
- $\tau = +[v_1, v_2, \ldots, v_n]$ (resp. $-[v_1, v_2, \ldots, v_n]$) and $\lambda_{g(v_i)}(v_i) = j_i$ for $i = 1, \ldots, n$.

we have then, with $\alpha := \alpha_+ - \alpha_-$ and $\beta := \beta_+ - \beta_-$, and with the same kind of proof as before (omitted here):

Theorem 6 *Let m, n and q be three positive integers and let M be an n-pseudomanifold. Let $\lambda = (\lambda_i)_{i=1,\ldots,q}$ be a q-multiple labeling, where for each i $\lambda_i : V(\mathsf{M}) \to \{-1,+1,-2,+2,\ldots,-m,+m\}$ is a labeling of the vertices of M such that there are no j_1, j_2 and $(v_1, v_2) \in E(\mathsf{M})$ such that $\lambda_{j_1}(v_1) = -\lambda_{j_2}(v_2)$.*

Then we have if M is oriented:

$$\sum_{1 \le j_1 < j_2 < \ldots < j_{n+1} \le m} \alpha(-j_1, +j_2, \ldots, (-1)^{n+1} j_{n+1}) + (-1)^n \alpha(+j_1, -j_2, \ldots, (-1)^n j_{n+1}) =$$

$$q \sum_{1 \le j_1 < j_2 < \ldots < j_n \le m} \beta(+j_1, -j_2, \ldots, (-1)^{n-1} j_n).$$

If M is not oriented, the formula holds modulo 2.

References

[1] R.B. Bapat, A constructive proof of a permutation-based generalization of Sperner's lemma, *Mathematical Programming*, 44:113-120, 1989.

[2] K. Fan, A generalization of Tucker's combinatorial lemma with topological applications, *Annals Math., II. Ser.*, 56:431-437, 1952.

[3] K. Fan, Simplicial maps from an orientable n-pseudomanifold into S^m with the octahedral triangulation, *J. Comb. Theory*, 2:588-602, 1967.

[4] D. Gale, Equilibrium in discrete exchange economy with money, *Inter. J. Game Theory*, 13:61-64, 1984.

[5] M. Kneser, Aufgabe 360, *Jahresbericht der Deutschen Mathematiker-Vereinigung, 2. Abteilung*, 58:27, 1955.

[6] H.W. Kuhn, A new proof of the fundamental theorem of algebra, *Mathematical Programming Study*, 1:148-158, 1974.

[7] S.-N. Lee and M.-H. Shih, A counting lemma and multiple combinatorial Stokes' theorem, *Eur. J. Combinatorics*, 19:969-979, 1998.

[8] J. Matousek, A combinatorial proof of Kneser's conjecture, *Combinatorica*, 24:163-170, 2004.

[9] J.Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Reading, MA, 1984.

[10] A. Schrijver, Vertex-critical subgraphs of Kneser graphs, *Nieuw Arch. Wiskd., III. Ser.*, 26:454-461, 1978.

[11] E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, *Abh. Math. Sem. Univ. Hamburg*, 6:265-272, 1928.

[12] A.W. Tucker, Some topological properties of disk and sphere, *Proc. of the First Canadian Mathematical Congress, Montreal 1945*, 285-309, 1946.

[13] G.M. Ziegler, Generalized Kneser coloring theorems with combinatorial proofs, *Inventiones Math.*, 147:671-691, 2002.