Topological Combinatorics – Ens Lyon

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CHAPTER 1

Preliminaries

1.1 Notations

For any positive integer a, the set $\{1, \ldots, a\}$ is denoted $[a]$. Let X be any set. Given a nonnegative integer k, we denote by $\binom{X}{k}$ the set of all subsets of cardinality k of X.

1.2 Complexity

A problem is *polynomial*, or *in P*, if there exists a polynomial time algorithm solving it.

A problem is *NP-hard* if unless $P \neq NP$, there is no polynomial algorithm solving it. The equality $P = NP$ is unlikely. Being NP-hard can thus be thought as being intrinsically hard.

1.3 Linear programming

1.3.1 Main results

A linear program is a mathematical program that can be written as

$$\begin{array}{rl} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} \geqslant \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{R}^n. \end{array}$$

where $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{c} \in \mathbb{R}^n$, and A is a $m \times n$ real matrix. This is the *inequation form.* It can equivalently be written under the *standard form*

$$\begin{array}{rl} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{R}^n_+ \end{array}$$

or the *canonical form*

$$\begin{array}{rl} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} \geqslant \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{R}^n_+. \end{array}$$

The set of feasible solutions of a linear program is a polyhedron.

Theorem 1.3.1. *Consider a minimization linear program.*

If it is feasible and bounded from below, then it has an optimal solution.

If it has an optimal solution, then it has an optimal solution that is a vertex of the polyhedron.

If it is feasible but is not bounded from below, then there is a ray $t \mapsto \boldsymbol{x}(t) = \boldsymbol{x}_0 + t\boldsymbol{q}$, *for some* $\boldsymbol{x}_0, \boldsymbol{q} \in \mathbb{R}^n$, *with* $\boldsymbol{x}(t)$ *feasible for all* $t \in \mathbb{R}_+$ *such that* $\lim_{t\to+\infty} \boldsymbol{c}^T\boldsymbol{x}(t) = -\infty$.

Proposition 1.3.2. *Consider a linear program in any form such that* $\boldsymbol{b}$ *and* A *have only rational coefficients. If it admis an optimal solution, then it has an optimal solution with rational coefficients.*

Proof. This is a consequence of the second point of Theorem 1.3.1. □

Note that there is no conditions on the coefficients of $\boldsymbol{c}$ in Proposition 1.3.2.

1.3.2 Bases

Consider the system

$$\begin{array}{rcl} A\boldsymbol{x} & = & \boldsymbol{b} \\ \boldsymbol{x} & \geqslant & \boldsymbol{0}. \end{array} \tag{P}$$

A is an $m \times n$ matrix of rank m. Given a subset X of $[n]$, we denote by A_X the matrix obtained by keeping from A only the columns indexed by elements of X.

A subset B of $[n]$ is a *basis* if A_B is nonsingular. Note that it implies that $|B| = m$. A basis is *feasible* if $A_B^{-1}\boldsymbol{b}$ is nonnegative. To each basis, there is an associated *basic solution* $\boldsymbol{y}$ defined by $\boldsymbol{y}_B = A_B^{-1}\boldsymbol{b}$ and $\boldsymbol{y}_{[n]\setminus B} = \boldsymbol{0}$. Note that if the basis is feasible, then the associated basic solution is feasible.

The pair $(A, \boldsymbol{b})$ is *generic* if for any $B \in \binom{[n]}{m}$, there is no $\boldsymbol{y} \in \mathbb{R}^B_+$ such that $A_B\boldsymbol{y} = \boldsymbol{b}$ and $\boldsymbol{y}$ has a component equal to zero.

Lemma 1.3.3. *Suppose* $(A, \boldsymbol{b})$ *generic and* $\{\boldsymbol{x} \in \mathbb{R}^n_+ : \ A\boldsymbol{x} = \boldsymbol{b}\}$ *bounded. Let* $B \in \binom{[n]}{m}$. *If* $A_B\boldsymbol{y} = \boldsymbol{b}$ *has a solution* $\boldsymbol{y}$ *in* $\mathbb{R}^B_+$, *then* A_B *is nonsingular (i.e.* B *is a feasible basis).*

Proof. Suppose for a contradiction that there is a $\boldsymbol{\lambda} \in \mathbb{R}_+^B \setminus \{\boldsymbol{0}\}$ such that $A_B\boldsymbol{\lambda} = \boldsymbol{0}$. Then, for any $t \in \mathbb{R}_+$, we have $A_B(\boldsymbol{y} + t\boldsymbol{\lambda}) = \boldsymbol{b}$. Since $\{\boldsymbol{x} \in \mathbb{R}_+^n : A\boldsymbol{x} = \boldsymbol{b}\}$ is bounded, there is necessarily a t such that $\boldsymbol{y} + t\boldsymbol{\lambda}$ has at least one component equal to 0, which contradicts the genericity assumption. □

Proposition 1.3.4. *Suppose* $(A, \boldsymbol{b})$ *generic and* $\{\boldsymbol{x} \in \mathbb{R}_+^n : \ A\boldsymbol{x} = \boldsymbol{b}\}$ *bounded. Let* $L \in \binom{[n]}{m+1}$. *The number of feasible bases contained in* L *is either* 0 *or* 2.

Proof. Let $Q = \{\boldsymbol{y} \in \mathbb{R}_+^L : \ A_L\boldsymbol{y} = \boldsymbol{b}\}$ and let $\boldsymbol{\lambda} \in \operatorname{Ker} A_L \setminus \{\boldsymbol{0}\}$. Assume that there is a feasible basis B in L. We are going to show that there is exactly one feasible basis in L distinct from B.

Let $\boldsymbol{x} \in \mathbb{R}_B^+$ be defined by $\boldsymbol{x}_B = A_B^{-1}\boldsymbol{b}$. Let $\boldsymbol{z} \in \mathbb{R}_L^+$ be defined by $z_i = x_i$ for $i \in B$ and $z_\ell = 0$, where ℓ is the unique element in $L \setminus B$. Note that we have $\lambda_\ell \neq 0$, otherwise A_B would have been singular. Without loss of generality, we assume that $\lambda_\ell > 0$. Defining $\boldsymbol{\mathcal{L}}$ to be the set of vectors $\boldsymbol{z} + t\boldsymbol{\lambda}$ for $t \in \mathbb{R}$, we have $Q = \boldsymbol{\mathcal{L}} \cap \mathbb{R}_+^L$. Since $\lambda_\ell > 0$, if $\boldsymbol{z} + t\boldsymbol{\lambda} \in Q$, then $t \geqslant 0$. When t goes to infinity, $\boldsymbol{z} + t\boldsymbol{\lambda}$ leaves Q for some value α because of the boundness assumption. Thus, we have $Q = \{\boldsymbol{z} + t\boldsymbol{\lambda} : \ t \in [0, \alpha]\}$.

Therefore, there are exactly two elements in Q having a support of size m: the one obtained for $t = 0$ and the one obtained for $t = \alpha$. Lemma 1.3.3 allows to conclude. □

1.4 Graphs

1.4.1 Basic definitions

A (undirected) *graph* is a pair $G = (V, E)$, where V is a finite set and E a finite family of unordered pairs uv from V. The elements of V are the *vertices* and the elements of E are the *edges*. When these sets have not been stated explicitly, we shall use $V(G)$ and $E(G)$ for the vertex set and the edge set of G, respectively.

We use the word 'family' rather than 'set' for the edges to indicate that some pairs of vertices may occur more than once as an edge.

Two vertices u and v are *adjacent* if uv is an edge. In such a case, u is a *neighbor* of v and vice-versa. The set of all neighbors of a vertex u is denoted $N(u)$. An edge uv is *incident* to both u and v. The set of edges incident to a vertex v is denoted $\delta(v)$. The *degree* of a vertex v, denoted $\deg(v)$, is the number of edges incident to it: $\deg(v) = |\delta(v)|$.

The following lemma, whose proof is left in exercise, will be useful.

Lemma 1.4.1. *The number of odd degree vertices in a graph is always even.*

A *path* is a sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_\ell, v_\ell$ where the v_i's are vertices, where the e_i's are edges, and where $e_i = v_{i-1}v_i$ for all i. A *circuit* is a path such that $v_0 = v_\ell$. A path (resp. circuit) is *elementary* if all e_i are distinct.

A graph is *connected* if there is a path between all pairs of vertices.

The *complete* graph K_t is the graph on t vertices with and edge between each pair of distinct vertices.

A subset S of vertices is *stable* if no two vertices in S are adjacent. The size of a largest stable set is denoted $\alpha(G)$. Determining its value is an NP-hard problem.

G is *complete* if for all $u, v \in V$ with $u \neq v$ we have $uv \in E$ (all possible edges are present).

1.4.2 Subgraphs

A graph $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. Given a subset X of V, we define $H[X]$ to be the graph $(X, E[X])$ where $E[X]$ is the set of all edges of E having both endpoints in X. Such a graph is a subgraph of G and is *induced* by X.

A *clique* is a complete subgraph. The size of the largest clique in G is denoted $\omega(G)$. Computing this quantity is an NP-hard problem.

1.4.3 Coloring

A *coloring* of a graph G is a map $c\colon V \to \mathbb{Z}_+$. The elements in $c(V)$ are the *colors*. The coloring is *proper* if no adjacent vertices get the same color. A proper coloring can also be seen as a partition of the vertex set into stable sets. The *chromatic number* is the smallest number k such that there is a proper coloring with k colors. It is denoted $\chi(G)$. Determining $\chi(G)$ is an NP-hard problem.

An (a, b)-*coloring* of G is a family of a stable sets such that each vertex is covered at least b times. Since it is a family, the same stable set can occur several times in an (a, b)-coloring. The *fractional chromatic number*, denoted $\chi_f(G)$, is the quantity

$$\inf\left\{\frac{a}{b} \colon \text{there exists a } (a, b)\text{-coloring of } G\right\}.$$

This infimum is actually a minimum as we are going to see. Since a proper coloring with a colors is an $(a, 1)$-coloring, we have

$$\chi_f(G) \leqslant \chi(G).$$

Alternatively, the fractional chromatic number of G is the optimal value of the following linear program:

$$\begin{aligned} \min \quad & \sum_{S\in\mathcal{S}} x_S \\ \text{s.t.} \quad & \sum_{S\in\mathcal{S}_v} x_S \geqslant 1 \quad && \forall v \in V \\ & x_S \in \mathbb{R}_+ && \forall S \in \mathcal{S}, \end{aligned}$$

where $\mathcal{S}$ is the set of all stable sets of G and $\mathcal{S}_v$ is the set of those containing v. The first point of Theorem 1.3.1 explains then why there is an (a, b)-coloring achieving the fractional chromatic number and thus why the infimum is actually a minimum in the definition of the fractional chromatic number.

There is an easy lower bound on the fractional chromatic number.

Proposition 1.4.2.

$$\frac{|V|}{\alpha(G)} \leqslant \chi_f(G).$$

Proof. Let $\mathcal{F}$ be an (a, b)-coloring. Denote by $\mathcal{F}_v$ the family of the stable sets in $\mathcal{F}$ containing v. We have

$$b|V| \leqslant \sum_{v\in V} |\mathcal{F}_v| = \sum_{S\in\mathcal{F}} |S| \leqslant a\alpha(G).$$

□

1.4.4 Graph homomorphism

Given two graphs G, H, a map $\phi\colon V(G) \to V(H)$ is a *graph homomorphism* from G to H if the following implication holds:

$$uv \text{ is an edge of } G \quad \Longrightarrow \quad \phi(u)\phi(v) \text{ is an edge of } H.$$

A proper coloring of a graph G with t colors is a graph homomorphism from G to the complete graph K_t. Since the composition of two graph homomorphisms is again a graph homomorphism, the existence of a graph homomorphism from G to H implies that the inequality $\chi(G) \leqslant \chi(H)$ holds.

1.4.5 Directed graphs

A *directed graph* is a pair $D = (V, A)$, where V is a finite set and A a finite family of ordered pairs (u, v) from V. The elements of V are the *vertices* and the elements of A are the *arcs*.

Two vertices u and v are *adjacent* if (u, v) is an arc. In such a case, u is an *inneighbor* of v and v an *outneighbor* of u. The set of all outneighbors (resp. inneighbors) of a vertex u is denoted $N^+(u)$ (resp. $N^-(u)$). An arc (u, v) *leaves* u and *enters* v. The set of arcs leaving (resp. entering) a vertex v is denoted $\delta^+(v)$ (resp. $\delta^-(v)$). The *outdegree* (resp. indegree) of a vertex v, denoted $\deg^+(v)$ (resp. $\deg^-(v)$), is the quantity $|\delta^+(v)|$ (resp. $|\delta^-(v)|$).

A *path* is a sequence $v_0, a_1, v_1, a_2, v_2, \ldots, a_\ell, v_\ell$ where the v_i's are vertices, where the a_i's are arcs, and where $a_i = (v_{i-1}, v_i)$ for all i. A *cycle* is a path such that $v_0 = v_\ell$. A path (resp. cycle) is *elementary* if all a_i are distinct.

A directed graph is *strongly connected* if for all $u, v \in V$, there is a path from u to v and a path from v to u.

1.5 Hypergraphs

A *hypergraph* $\mathcal{H}$ is a pair $(V, \mathcal{E})$ with $\mathcal{E} \subseteq 2^V$. The elements in V are the vertices of $\mathcal{H}$ and those in $\mathcal{E}$ are its edges. A *matching* of $\mathcal{H}$ is a collection of pairwise disjoint edges. The *matching number* of $\mathcal{H}$, denoted by $\nu(\mathcal{H})$, is the maximum size of a matching. A *vertex cover* of $\mathcal{H}$ is a subset C of V such that each edge of $\mathcal{H}$ contains at least one element from U. The *vertex cover number* of $\mathcal{H}$, denoted by $\tau(\mathcal{H})$, is the minimum size of a vertex cover.

The *fractional matching number* of $\mathcal{H}$, denoted by $\nu^*(\mathcal{H})$, is the optimal value of the following linear program.

$$\begin{array}{lll} \max & \displaystyle\sum_{e\in\mathcal{E}} x_e & \\ \text{s.t.} & \displaystyle\sum_{e\in\delta(v)} x_e \leqslant 1 & \forall v \in V \\ & x_e \in \mathbb{R}_+ & \forall e \in \mathcal{E}, \end{array}$$

where $\delta(v)$ is the set of edges containing v.

The *fractional vertex cover number* of $\mathcal{H}$, denoted by $\tau^*(\mathcal{H})$, is the optimal value of the following linear program.

$$\begin{array}{lll} \min & \displaystyle\sum_{v\in V} y_v & \\ \text{s.t.} & \displaystyle\sum_{v\in e} y_v \geqslant 1 & \forall e \in \mathcal{E} \\ & y_v \in \mathbb{R}_+ & \forall v \in V. \end{array}$$

Since these two linear programs are dual to each other, we have

$$\nu(\mathcal{H}) \leqslant \nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \leqslant \tau(\mathcal{H}).$$

More surprisingly, the matching number can also be bounded from below by the fractional matching number.

Theorem 1.5.1 (Füredi, 1981)**.** *If all edges of $\mathcal{H}$ are of size at most d, then $\nu(\mathcal{H}) \geqslant \frac{\nu^*(\mathcal{H})}{d-1+\frac{1}{d}}$.*

1.6 Polyhedra

A *polyhedron* P of $\mathbb{R}^d$ is the intersection of finitely many closed halfspaces. A point v of P is a *vertex*, or an *extreme point*, of P if any segment of P containing v in its interior is v itself. The set of all vertices is denoted $V(P)$. The intersection of a polyhedron with a closed half-space whose boundary avoids its relative interior is a *face*. A face is also a polyhedron. There is a unique face of maximal dimension: the polyhedron itself. A *facet* is a face of maximal dimension minus one. A vertex is a 0-dimensional face. $\varnothing$ is the unique face of dimension -1.

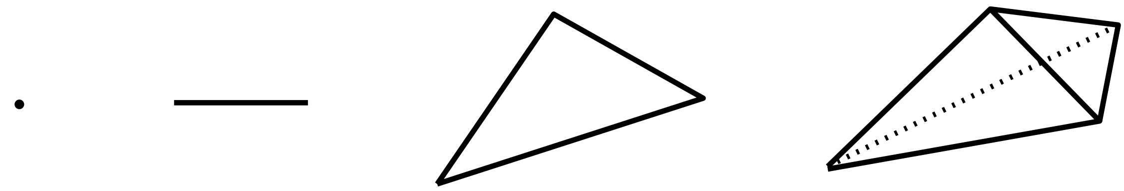
A *polytope* is a bounded polyhedron. It can also be described as the convex hull of finitely many points. In particular, any polytope is the convex hull of its vertices. A *simplex* is the convex hull of affinely independent points. It is a polytope. By definition, all faces of a simplex are again simplices. The dimension of a simplex is also the number of its vertices minus one. Simplices in dimension 0, 1, 2, and 3 are depicted in Figure 1.1. For these values, the simplices have special names: a 1-dimensional simplex in an *edge*, a 2-dimensional simplex is a *triangle*, and a 3-dimensional simplex is a *tetrahedron*.

A *polyhedral cone* is a polyhedron C such that for any $\boldsymbol{y} \in C$ and any $\lambda \in \mathbb{R}_+$, we have $\lambda \boldsymbol{y} \in C$. The *extreme rays* of C are the ray originating at $\mathbf{0}$, contained in C, and not convex combination of any other rays in C. For any polyhedron P, there exists a polytope Q and a polyhedral cone C such that any $\boldsymbol{x} \in P$ can be written as $\boldsymbol{y}+\boldsymbol{z}$, where $\boldsymbol{y} \in Q$ and $\boldsymbol{z} \in C$. This is the Minkowski-Weyl theorem. Such a polyhedral cone C is uniquely determined by P (it is not the case for the polytope Q). The *extreme rays* of P are the extreme rays of C.

1.7 Simplicial complexes

A *simplicial complex* K is a collection of simplices satisfying the following properties:

- if τ is a face of a simplex $\sigma \in \mathsf{K}$, then $\tau \in \mathsf{K}$.



$d = 0$ $d = 1$ $d = 2$ $d = 3$

Figure 1.1: d-dimensional simplices for $d = 0, 1, 2, 3$

- if σ and σ' are two simplices of K, then their intersection $\sigma \cap \sigma'$ is a face of both (in particular, this intersection can be empty).

Its dimension – denoted $\dim \mathsf{K}$ – is $\max_{\sigma\in\mathsf{K}} \dim \sigma$. Its vertex set is the set of its 0-dimensional simplices: $V(\mathsf{K}) = \bigcup_{\sigma\in\mathsf{K}} V(\sigma)$. The set of its edges is denoted $E(\mathsf{K})$ and is the set of its 1-dimensional simplices.

Figure 1.2 shows a simplex of dimension 2, whereas Figure 1.3 shows a collection of simplices that do not provide a simplicial complex (the intersection of the two triangles is a face of none of them).

The *k-skeleton* of a simplicial complex K, denoted $\mathsf{K}^{(k)}$, is the simplicial complex obtained from K by removing all simplices of dimension greater than k.

The union $\bigcup_{\sigma\in\mathsf{K}} \sigma$ of all simplices of a simplicial complex K is called the *polyhedron* of the simplicial complex. It is denoted $\|\mathsf{K}\|$.

Let X be a topological space. A simplicial complex T is a *triangulation* of X if $\|\mathsf{T}\|$ is homeomorphic to X.

1.8 Barycentric subdivision and posets

The barycentric subdivision of a simplicial complex K is defined as

$$\mathrm{sd}(\mathsf{K}) = \{\mathrm{conv}(\boldsymbol{b}_{\sigma_0}, \ldots, \boldsymbol{b}_{\sigma_k}) \colon \sigma_0, \ldots, \sigma_k \in \mathsf{K} \setminus \{\varnothing\},\ \sigma_0 \subset \cdots \subset \sigma_k\},$$

where $\boldsymbol{b}_\sigma$ denotes the barycenter of a simplex σ. The fact that $\mathrm{sd}(\mathsf{K})$ is indeed a simplicial complex deserves a proof.

Proposition 1.8.1. $\mathrm{sd}(\mathsf{K})$ *is a simplicial complex.*

Proof. `To be completed` □

The following proposition, whose proof is left in exercise, is useful to obtain triangulation of arbitrarily small mesh.

Proposition 1.8.2.

$$\mathrm{diam}(\mathrm{sd}\,\mathsf{K}) < \frac{\dim \mathsf{K}}{1 + \dim \mathsf{K}} \mathrm{diam}(\mathsf{K}).$$

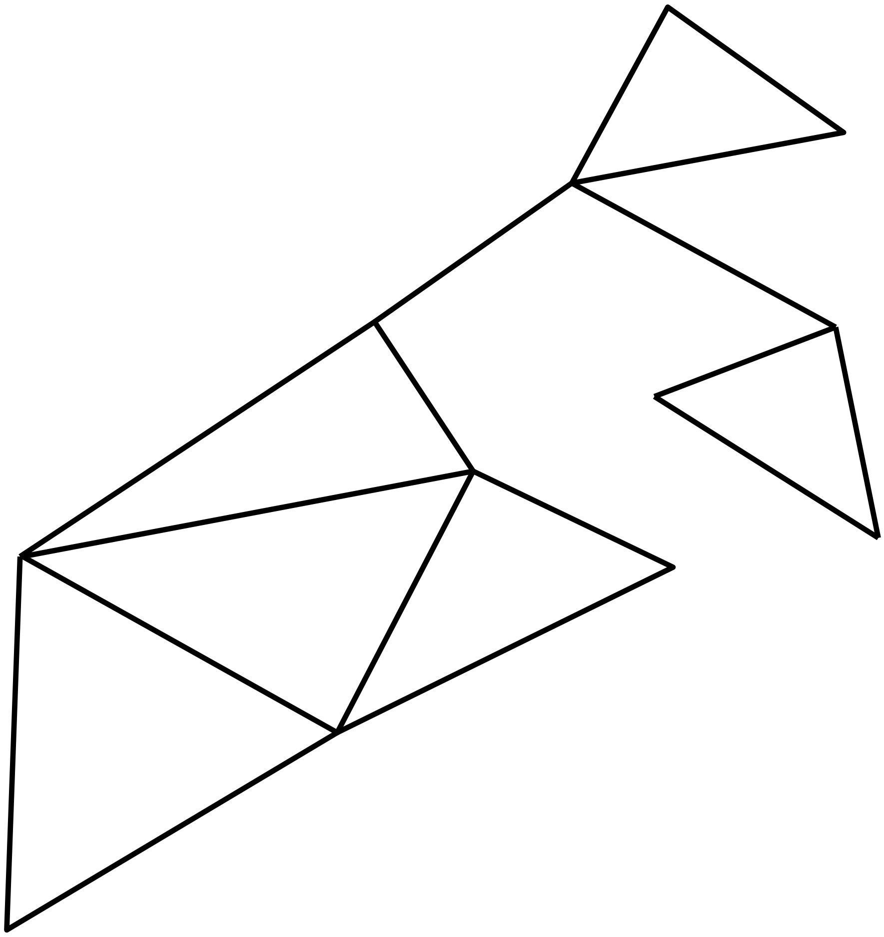


Figure 1.2: A simplicial complex

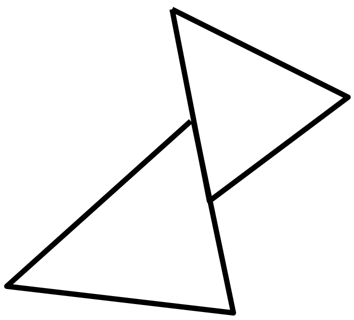


Figure 1.3: Not a simplicial complex

1.9 Signed vectors

A useful tool in combinatorics is the *signed vectors*. A signed vector is an element of $\{+, -, 0\}^n$ for some positive n.

The partial order given by

$$0 \prec +, \; 0 \prec -, \quad \text{and} \quad + \text{ and } - \text{ not comparable}$$

is extended on signed vectors by taking the product order.

Given a signed vector $\boldsymbol{x} = (x_1, \ldots, x_n)$, we denote by $\boldsymbol{x}^+$ (resp. $\boldsymbol{x}^-$) the set of indices such that $x_i = +$ ($x_i = -$). We always have $\boldsymbol{x}^+ \cap \boldsymbol{x}^- = \varnothing$.

CHAPTER 2

Sperner's lemma, Tucker's lemma, and their relatives

2.1 Sperner's lemma

Sperner's lemma is an important result in combinatorial topology. It was originally proposed by Sperner Sperner (1928) to obtain a simple and constructive proof of Brouwer's fixed-point theorem stating that any continuous map from a finite-dimensional ball into itself has a fixed-point. Brouwer's fixed-point theorem has numerous applications in mathematics and economy. The relation between Sperner's lemma and Brouwer's theorem can be found for instance in the recent book by De Longueville De Longueville (2012). The original proof by Sperner, even constructive, was not algorithmic. Motivated by concrete applications, Scarf, inspired by the Lemke algorithm, proposed later an algorithmic proof Scarf (1982), which is actually a pivot-based one. It is an adaptation of his proof that is given below.

Sperner's lemma itself has many applications, in game theory or in combinatorics. The application to cake cutting developed below is such an example. Another example is a result by Aharoni and Haxell Aharoni and Haxell (2000), who prove a very general sufficient condition for the existence of a system of distinct representatives for a family of hypergraphs. Another nice application of the Sperner lemma is a theorem due to Monsky Monsky (1970): any dissection of a rectangle into triangles of same area needs an even number of triangles.

One of the multiple versions of Sperner's lemma is the following theorem, proposed by Scarf Scarf (1967).

Theorem 2.1.1 (Sperner's lemma)**.** *Let* T *be a triangulation of the standard* d*-dimensional simplex* $\triangle^d$*. Let* $\lambda\colon V(\mathsf{T}) \to [d+1]$ *be a labeling of the vertices of* T *such that*

- *each vertex of* $\triangle^d$ *gets a distinct label,*

- *each vertex v of T gets a label in $\lambda(V(F))$, where F is the minimal face of $\triangle^d$ containing v.*

Then there is an odd number of simplices $\sigma \in \mathsf{T}$ such that $\lambda(V(\sigma)) = [d+1]$.

A simplex σ of T is *fully labeled* if $\lambda(\sigma) = [d+1]$. Sperner's lemma states that under the conditions given on the labeling, there exists always an odd number of fully-labeled simplices.

Proof of Theorem 2.1.1. The proof works by induction on d. If $d = 0$, there is nothing to prove.

If $d > 0$, we build a graph G whose vertices are the d-dimensional simplices, with an additional "dummy" vertex r. Two vertices are connected by an edge if the corresponding d-simplices share a facet τ such that $\lambda(V(\tau)) = [d]$. Moreover, a vertex v of G is connected to r if the simplex corresponding to v has a facet τ on the boundary of $\triangle^d$ such that $\lambda(V(\tau)) = [d]$.

The degree of a vertex $v \neq r$ is different from 0 only if the corresponding simplex has a facet τ such that $\lambda(V(\tau)) = [d]$. This degree is always 2 except if the simplex is fully-labeled, in which case the degree is 1. By induction applied on the facet of $\triangle^d$ using labels $1, \ldots, d$,we get that the degree of r is odd. There is thus another odd degree vertex (Lemma 1.4.1), which necessarily corresponds to a fully-labeled simplex. $\square$

Sperner's lemma can be used to prove Brouwer's fixed-point theorem.

2.2 Surjectivity of continuous self-maps stabilizing the faces of a polytope

Brouwer's theorem can be used to prove the following useful topological fact.

Lemma 2.2.1. *Let P be a polytope and f a continuous self-map of P such that $f(F) \subseteq F$ for each face F of P. Then f is surjective.*

Proof. We proceed by induction on the dimension d of the polytope. For $d = 0$, the theorem is obviously true.

Consider $d > 0$ and suppose for a contradiction that there exists a point $\boldsymbol{y} \in P$ that is not in the image of f. By induction, we know that $\boldsymbol{y}$ is in the interior of P. For any point $\boldsymbol{x}' \in P \setminus \{\boldsymbol{y}\}$, the half-ray originating at $\boldsymbol{x}'$ and passing through $\boldsymbol{y}$ is well-defined. It has a unique intersection with ∂P – the boundary of P. This defines a continuous map $\rho \colon P \setminus \{\boldsymbol{y}\} \to \partial P$. Now, apply Brouwer's theorem to the map $\rho \circ f$. There is a point $\boldsymbol{z}$ that is fixed by this map. By definition of ρ, the point $\boldsymbol{z}$ is on a proper face F of P that does not contain $f(\boldsymbol{z})$, a contradiction. $\square$

2.3 Shapley's lemma

A hypergraph $\mathcal{H}$ is *balanced* if there exists a weight function $w\colon E(\mathcal{H}) \to \mathbb{R}_+$ such that $w(\delta(v)) = 1$ for every vertex v of $\mathcal{H}$.

Theorem 2.3.1 (Shapley's lemma)**.** *Let* T *be a triangulation of the standard* d*-dimensional simplex* $\triangle^d = \{\boldsymbol{x} \in \mathbb{R}_+^{d+1} \colon \sum_{i=1}^{d+1} x_i = 1\}$. *Let* $\lambda\colon V(\mathsf{T}) \to 2^{[d+1]} \setminus \{\varnothing\}$ *be a labeling of the vertices of* T *with nonempty subsets of* $[d+1]$ *such that for each vertex* v *in* $V(\mathsf{T})$*, we have* $\lambda(v) \subseteq \operatorname{supp}(v)$*. Then there is a simplex* $\sigma \in \mathsf{T}$ *such that the hypergraph* $([d+1], \lambda(V(\sigma)))$ *is balanced.*

Proof. We interpret λ as a simplicial map from T to $\operatorname{sd} \triangle^d$, which in turn can be seen as a continuous self-map of $\triangle^d$, which stabilizes the faces. According to Lemma 2.2.1, there is a point $\boldsymbol{x}_0 \in \triangle^d$ such that $\lambda(\boldsymbol{x}_0) = (\frac{1}{d+1}, \frac{1}{d+1}, \ldots, \frac{1}{d+1})$. The point $\boldsymbol{x}_0$ is contained in a d-simplex σ of T. The labels of the vertices of σ form the edges of a balanced hypergraph (the weight function being given by the barycentric coordinates of $\boldsymbol{x}_0$ in σ). □

2.4 Ky Fan's lemma

2.4.1 The general Ky Fan's lemma

The following theorem is due to Ky Fan Fan (1952).

Theorem 2.4.1. *Suppose that* T *is a triangulation of the* d*-dimensional sphere* S^d *such that if* $\sigma \in \mathsf{T}$ *then* $-\sigma \in \mathsf{T}$.

Let $\lambda\colon V(\mathsf{T}) \to \{-1, +1, \ldots, -m, +m\}$ *be a labeling of the vertices such that*

- $\lambda(-v) = -\lambda(v)$ *for each* $v \in V(\mathsf{T})$
- $\lambda(u) + \lambda(v) \neq 0$ *for each edge* $uv \in E(\mathsf{T})$.

Then there is an odd number of d*-simplices* σ *of* T *such that* $\lambda(V(\sigma))$ *has the form*

$$\{-j_0, +j_1, \ldots, (-1)^{d+1} j_d\}$$

with $j_0 < j_1 < \cdots < j_d$*. In particular, we have* $m \geqslant d+1$.

The special case with $m = d$, which states that in this case a map with the required conditions cannot exist, is known as Tucker's lemma. Tucker proved its special case for $d = 2$ in 1946 Tucker (1946). Tucker's lemma can be used to prove the Borsuk-Ulam theorem.

To author's knowledge, there is no combinatorial proof for this version. We propose a proof for a triangulation with an additional property we describe now.

We see S^d as being $\{(x_1,\ldots,x_{d+1})\in\mathbb{R}^{d+1}\colon \sum_{i=1}^{d+1}x_i^2=1\}$ and decompose it into hemispheres: $H_0^+,H_0^-,H_1^+,H_1^-,\ldots,H_d^+,H_d^-$. The hemisphere H_d^+ is the set of points in S^d with $x_{d+1}\geqslant 0$ and the hemisphere H_d^- is the set of points in S^d with $x_{d+1}\leqslant 0$. We recursively define the other hemispheres similarly on the equator of S^d. The additional property we require for the triangulation is that it induces a triangulation of the hemispheres (if the interior of a simplex of T iintersects one of the hemispheres, then it is fully contained in the hemisphere).

Proof of Theorem 2.4.1 for the special triangulation. We proceed by induction on d. The theorem is obviously true for $d=0$. Let us now suppose that $d>0$. For $\varepsilon\in\{-,+\}$, we say that a k-simplex σ is *ε-alternating* if $\lambda(V(\sigma))$ has the form $\{\varepsilon j_0,-\varepsilon j_1,\ldots,(-1)^k\varepsilon j_k\}$ with $j_0<j_1<\cdots<j_k$. We say that it is *almost ε-alternating* it it has an ε-alternating facet without being itself ε-alternating.

We build a graph whose vertices are all d-simplices of H_d^+, with an additional dummy vertex r. An edge links two vertices if the corresponding simplices share a common alternating facet. There is also an edge between r and another vertex if this latter corresponds to a d-simplex having an $-$-alternating facet on the boundary of H_d^+. By induction, the degree of r is odd. There is thus an additional odd number of odd degree vertices in this graph. They are exactly the alternating d-simplices of H_d^+. There is thus an odd number of alternating simplices in H_d^+. Using the symmetry, there is an odd number of $-$-alternating simplices on S^d. □

2.4.2 Ky Fan's lemma for signed vectors

The following theorem is a corollary of Theorem 2.4.1.

Theorem 2.4.2. *Let $\lambda:\{+,-,0\}^n\setminus\{(0,\ldots,0)\}\to\{-1,+1,\ldots,-m,+m\}$ be a map satisfying the following two properties:*

- $\lambda(-\boldsymbol{x})=-\lambda(\boldsymbol{x})$ *for all* $\boldsymbol{x}$
- $\lambda(\boldsymbol{x})+\lambda(\boldsymbol{y})\neq 0$ *for all* $\boldsymbol{x}\preccurlyeq\boldsymbol{y}$.

There there is an odd number of chains

$$\boldsymbol{x}^{(1)}\prec\boldsymbol{x}^{(2)}\prec\cdots\prec\boldsymbol{x}^{(n)}$$

such that $\lambda(\{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}\})$ *has the form*

$$\{-j_1, +j_2, \ldots, (-1)^n j_n\}$$

with $j_1 < j_2 < \cdots < j_n$. *In particular, we have* $m \geqslant n$.

Proof. There is a one-to-one correspondence between signed vectors and the barycenters of the proper faces of the $(d+1)$-dimensional cube. It is actually a one-to-one correspondence between the chains and the simplices of the barycentric subdivision of the boundary of the $(d+1)$-dimensional cube. □

CHAPTER 3

The Borsuk–Ulam theorem

3.1 Introduction

The most useful result from topology, in terms of its applications to combinatorics, is arguably the Borsuk–Ulam theorem, conjectured by Ulam and proved by Borsuk in 1933.

Theorem 3.1.1. *If $f : S^d \to \mathbb{R}^d$ is a continuous map, then there exists $x \in S^d$ such that $f(x) = f(-x)$.*

A useful way to picture the $d = 2$ case is to imagine deflating a ball and lying it flat on the floor. Then some points of the ball which were initially antipodal will be lying on top of each other. Another popular example is the fact that at any point in time, there exist two antipodal points on the surface of the earth with the same temperature and pressure.

The Borsuk–Ulam theorem comes in many guises, some of which we shall now give.

A $\mathbb{Z}_2$-space is a pair (X, ν), where X is a space and $\nu : X \to X$ is a homeomorphism such that $\nu \circ \nu = \mathrm{id}$. A standard example of a $\mathbb{Z}_2$-space is the sphere S^d equipped with the antipodality map $\nu : x \mapsto -x$. Suppose (X, ν) and (Y, ξ) are two $\mathbb{Z}_2$-spaces. A map $f : X \to Y$ is said to be *equivariant* if $f(\nu(x)) = \xi(f(x))$ for all $x \in X$.

Theorem 3.1.2. *There exists no continuous equivariant map $f : S^d \to S^{d-1}$ (that is, a map such that $f(-x) = -f(x)$).*

The following version was proved by Lyusternik and Schnirelmann *before* Borsuk's proof.

Theorem 3.1.3. *If the sphere S^d is covered by $d+1$ closed sets $F_1, \ldots, F_{d+1}$, then there exists an $i \in \{1, \ldots, d+1\}$ such that $F_i \cap (-F_i) \neq \varnothing$.*

The statement is also true for open covers of the sphere:

Theorem 3.1.4. *If the sphere S^d is covered by $d+1$ open sets $U_1, \ldots, U_{d+1}$, then there exists an $i \in \{1, \ldots, d+1\}$ such that $U_i \cap (-U_i) \neq \varnothing$.*

Proving the equivalence of these statements is left as an exercise.

We conclude this section by deducing Brouwer's fixed point theorem from Theorem 3.1.2.

Proof of Brouwer's fixed point theorem. Suppose that $f : B^d \to B^d$ is continuous and has no fixed point. For every $x \in B^d$, define $g(x)$ as the intersection of the boundary S^{d-1} and the ray originating from $f(x)$ and passing through x. Let $\pi : S^d \to B^d$ be the projection $\pi : (x_1, x_2, \ldots, x_{d+1}) \mapsto (x_1, x_2, \ldots, x_d)$. For $x \in S^d$, set

$$h(x) = \begin{cases} g(\pi(x)) & \text{if } x_d \geqslant 0 \\ -g(\pi(-x)) & \text{if } x_d < 0. \end{cases}$$

This defines a map $h : S^d \to S^{d-1}$. Now, consider any pair $x, -x$ of antipodal points. Without loss of generality, assume $x_d \geqslant 0$, so $-x_d \leqslant 0$. We have $h(x) + h(-x) = g(\pi(x)) - g(\pi(x)) = 0$, so h is equivariant. This contradicts Theorem 3.1.2. □

3.2 The ham sandwich theorem

Theorem 3.2.1. *Let $\mu_1, \mu_2, \ldots, \mu_d$ be finite Borel measures on $\mathbb{R}^d$ such that every hyperplane has measure 0 for each of the μ_i. Then there exists a hyperplane h such that*

$$\mu_i(h^+) = \mu_i(h^-) = \tfrac{1}{2}\mu_i(\mathbb{R}^d) \text{ for } i = 1, 2, \ldots, d,$$

where h^+ and h^- are the two half-spaces defined by h.

Proof. TO DO. □

For combinatorial applications, the following discrete version of the ham sandwich theorem is more useful.

Theorem 3.2.2. *Let $A_1, A_2, \ldots, A_d \subset \mathbb{R}^d$ be finite point sets. Then there exists a hyperplane h that simultaneously bisects $A_1, A_2, \ldots, A_d$.*

Proof. The idea of the proof is to replace the points of A_i by tiny balls and then apply Theorem 3.2.1.

TODO □

3.3 Borsuk graphs

Recall that the chromatic number of a graph is the smallest number of colours such that the vertices can be coloured so that adjacent vertices receive different colours.

It is easy to see that the chromatic number is bounded from below by the clique number, the size of the largest complete subgraph: $\chi(G) \geqslant \omega(G)$. It is natural to ask whether χ can be bounded from above by some function of ω. This is not the case, because there exist triangle-free graphs of arbitrarily high chromatic number. This was first shown constructively by Zykov. The Borsuk–Ulam theorem provides an alternative proof, as noted by Erds and Hajnal Erdös and Hajnal (1967).

They defined the *Borsuk graph* $BG(n, \alpha)$ as the (infinite) graph whose vertices are the points of $\mathbb{R}^{n+1}$ on S^n, and the edges connect points at Euclidean distance at least α, where $0 < \alpha < 2$.

Theorem 3.3.1. $\chi(BG(d, \alpha)) \geqslant d + 2$.

Indeed, it can be shown that this theorem is equivalent to the Borsuk–Ulam theorem.

CHAPTER 4

Kneser graphs

4.1 Introduction

For positive integers n, k such that $n \geqslant 2k$, we define the *Kneser graph* $\mathrm{KG}(n,k) = (V,E)$ by

$$\begin{array}{rcl} V & = & \binom{[n]}{k} \\ E & = & \{\{e,f\} : e,f \in V,\, e \cap f = \emptyset\}. \end{array}$$

Kneser graphs have been extensively studied, see **??** among many references. Martine Kneser **?** conjectured the following result, which was finally confirmed by Lovász's theorem **?**.

Theorem 4.1.1 (Lovász theorem)**.** *Given n and k two positive integers with $n \geqslant 2k$, we have $\chi(\mathrm{KG}(n,k)) = n - 2k + 2$.*

Lovász actually proved more than this theorem and gave a general lower bound on the chromatic number of a graph in terms of the connectivity of a simplicial complex associated to the graph. One of his main tool was the Borsuk-Ulam theorem and his paper is often seen as the first application of algebraic topology to combinatorics.

Kneser graphs have received a great attention not only because of the beauty of Kneser's conjecture but also because they enjoy many interesting properties. One of them is an arbitrary gap between their fractional chromatic number and the chromatic number.

4.2 Fractional chromatic number

Theorem 4.2.1. *Given n and k two positive integers with $n \geqslant 2k$, we have $\chi_f(\mathrm{KG}(n,k)) = \dfrac{n}{k}$.*

Proof. The k-sets containing a given fixed element $i \in [n]$ form a stable set of size $\binom{n-1}{r-1}$, and every vertex lies in precisely k of these stable sets. For each stable set $S \in \mathcal{S}$ of this form, let $x_S = 1/k$, and let $x_S = 0$ for the other stable sets. This defines a fractional colouring of $\mathrm{KG}(n,k)$ of weight n/k, so $\chi_f(\mathrm{KG}(n,k)) \leqslant \frac{n}{k}$.

By the Erdös–Ko–Rado theorem, $\alpha(\mathrm{KG}(n,k)) = \binom{n-1}{k-1}$. Hence,

$$\chi_f(\mathrm{KG}(n,k)) \geqslant \frac{|V(\mathrm{KG}(n,k))|}{\alpha(G)} = \frac{\binom{n}{k}}{\binom{n-1}{k-1}} = \frac{n}{k}.$$

□

4.3 Chromatic number

The fact that $\chi(\mathrm{KG}(n,k)) \leqslant n-2k+2$ was already known by Kneser and can be proved by the following explicit coloring

$$c : A \in \binom{[n]}{k} \to \min(\min A, n-2k+2) \in [n-2k+2].$$

Proof of Theorem 4.1.1. It is enough to prove that $\chi(\mathrm{KG}(n,k)) \geqslant n-2k+2$. Let c be a proper coloring of $\mathrm{KG}(n,k)$ with t colors.

Let $m = t+2k-2$. We define a map $\lambda : \{+,-,0\}^n \setminus \{(0,\ldots,0)\} \to \{-1,+1,\ldots,-m,+m\}$ as follows. Let $\boldsymbol{x} \in \{+,-,0\}^n \setminus \{(0,\ldots,0)\}$.

If $|\boldsymbol{x}^+| + |\boldsymbol{x}^-| \leqslant 2k-2$, we define $\lambda(\boldsymbol{x})$ to be $\varepsilon(|\boldsymbol{x}^+| + |\boldsymbol{x}^-|)$, where $\varepsilon = \min_{i\in[n]}\{x_i : x_i \neq 0\}$ (first nonzero entry of $\boldsymbol{x}$).

If $|\boldsymbol{x}^+| + |\boldsymbol{x}^-| \geqslant 2k-1$, we define $\lambda(\boldsymbol{x})$ to be $\varepsilon(2k-2+\min(a^+,a^-))$, where

$$a^+ = \min\{c(S) : S \subseteq \boldsymbol{x}^+ \text{ and } |S| = k\} \quad \text{and} \quad a^- = \min\{c(S) : S \subseteq \boldsymbol{x}^- \text{ and } |S| = k\}$$

and where $\varepsilon = +$ if $a^+ < a^-$ and $\varepsilon = -$ otherwise.

It can be checked that λ satisfies the condition of Theorem 2.4.2. We have thus $t+2k-2 \geqslant n$, which implies the desired inequality. □

CHAPTER 5

Mycielski graphs

5.1 Introduction

The Mycielski construction Mycielski (1955) is one of the earliest and arguably simplest constructions of triangle-free graphs of arbitrary chromatic number. Given a graph $G = (V, E)$, we let $M_2(G)$ be the graph with vertex set $V \times \{0, 1\} \cup \{z\}$, where there is an edge $\{(u, 0), (v, 0)\}$ and $\{(u, 0), (v, 1)\}$ whenever $\{u, v\} \in E$, and an edge $\{(u, 1), z\}$ for all $u \in V$ (see Figure 5.1). It is an easy exercise to show that the chromatic number increases with each iteration of $M_2(\cdot)$.

The construction was generalised by Stiebitz (1985) (see also Sachs and Stiebitz (1989), Gyárfás et al. (2004)), and independently by Van Ngoc (1987) (see also Van Ngoc and Tuza (1995)), in the following way. Given a graph $G = (V, E)$ and an integer $r \geqslant 1$, we define $M_r(G)$ as the graph with vertex set $V \times \{0, \ldots, r-1\} \cup \{z\}$, where there is an edge $\{(u, 0), (v, 0)\}$ and $\{(u, i), (v, i+1)\}$ whenever $\{u, v\} \in E$, and an edge $\{(u, r-1), z\}$ for all $u \in V$. The construction is illustrated in Figure 5.2.

If $r > 2$, it is no longer true that the chromatic number increases with each iteration of $M_r(\cdot)$. For instance, it can be shown that if $\overline{C_7}$ is the complement of the 7-cycle, then $\chi(M_3(\overline{C_7})) = \chi(\overline{C_7}) = 4$. However, Stiebitz Stiebitz (1985) was able to show that the chromatic does increase with each iteration

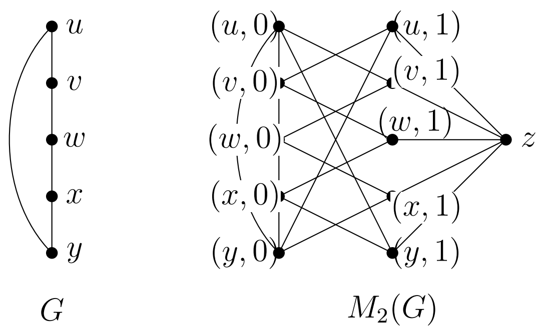


Figure 5.1: The Mycielski construction $M_2(\cdot)$

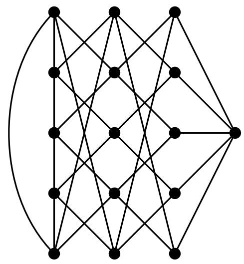


Figure 5.2: The graph $M_3(C_5) \cong M_3(M_2(K_2)) \in \mathcal{M}_4$.

of $M_r(\cdot)$ if we start with an odd cycle, or some other suitably chosen graph. For every integer $k \geqslant 2$, let us denote by $\mathcal{M}_k$ the set of all 'generalised Mycielski graphs' obtained from K_2 by $k-2$ iterations of $M_r(\cdot)$, where the value of r can vary from iteration to iteration. Stiebitz (1985) (see also Gyárfás et al. (2004), Matoušek (2003)) proved the following.

Theorem 5.1.1 (Stiebitz (1985))**.** *If $G \in \mathcal{M}_k$, then $\chi(G) \geqslant k$.*

Stiebitz's proof uses the neighbourhood complex defined by Lovász. Namely, Stiebitz proved the following lemma.

Lemma 5.1.2. *For every graph G and every $r \geqslant 2$, the neighbourhood complex of $M_r(G)$ is homotopy equivalent to the suspension* $\mathrm{susp}(N(G))$.

The neighbourhood complex of K_2 is homotopy equivalent (in fact, homeomorphic) to S^0, so after $k-2$ iterations of $M_r(\cdot)$ (where the value of r can vary from iteration to iteration) the neighbourhood complex is homotopy equivalent to S^{k-2}, and in particular, it is $(k-3)$-connected. Hence, by Theorem **??**, the chromatic number is k. This proves Theorem 5.1.1.

CHAPTER 6

Fair divisions

In many situations, goods have to be divided between players in a "fair" way. Fair can have several different meanings. The division can be fair if each player has the feeling to have at least as much as any other player. The division can also be fair when the division is made by a board of examiners and the division is fair according to each examiner's evaluation. And so on.

In the present chapter, we present two families of result with distinct definition of fairness, namely cake-cutting and necklace-splitting. These results are fascinating by many aspects.

6.1 Cake cutting

In this section, we focus on the following problem: cut a cake in such a way each player has the feeling to get at least as much as any other player. We will see that under natural assumption, it is possible to cut a cake among n players with $n-1$ cuts and get such a division, which is said to be *envy-free*. It is a theorem due to Stromquist. The main tool for proving this result is Sperner's lemma.

According to Su Su (1999), Forest Simmons found a constructive proof of Stromquist's theorem based on Sperner's lemma Sperner (1928), the combinatorial counterpart of Brouwer's fixed point theorem.

Each cake is identified with the interval $[0,1]$. A *division* of the cake into r pieces is an r-tuple $\boldsymbol{x} = (x_1,\ldots,x_r)$, with $x_j \geqslant 0$ for all $j \in [r]$ and $\sum_{j=1}^r x_j = 1$, where x_j is the size of the jth piece (ordered from left to right) of the cake. Given a division $\boldsymbol{x}$ of the cake into r pieces, a player *prefers* a certain piece if that player does not think that any other piece is strictly better. For some divisions a player may be indifferent to two or more "preferred" pieces.

The following assumptions on the preferences are the ones considered by Stromquist.

1. *Independence of preferences:* The preferences of one player do not de-

pend of the choices made by the other players.

2. *The players are hungry:* A player will never choose an empty piece.

3. *Preference sets are closed:* If one player prefer the same piece for a convergent sequence of division, then that piece will be preferred at the limit.

Theorem 6.1.1. *Assume that r players want to divide the cake.There always exists an envy-free division into r pieces.*

The proof is based on Sperner's lemma and uses actually two distinct labelings. Let T be a triangulation of the standard simplex $\triangle^{r-1}$. A labeling $\mu : V(\mathsf{T}) \to [r]$ is an *owner-labeling* if each $(r-1)$-simplex σ gets all labels on its vertices: $\mu(V(\sigma)) = [r]$. Not all triangulations admit an owner-labeling.

Lemma 6.1.2. *For all $\varepsilon > 0$, there exists a triangulation of the standard simplex admitting an owner-labeling and such that* $\operatorname{diam} \mathsf{T} < \varepsilon$*.*

Proof. Indeed, the barycentric subdivision of any triangulation of the standard simplex admits an owner-labeling. □

Proof of Theorem 6.1.1. For each point $\boldsymbol{x} \in \triangle^{r-1}$, we define $\nu_i(\boldsymbol{x})$ to be the piece preferred by player i when the division of the cake is $\boldsymbol{x}$ (in case of a tie, make an arbitrary choice).

We take T a triangulation of the standard simplex $\triangle^{r-1}$ with an owner-labeling μ and we suppose T to be of arbitrary small mesh size. It exists according to Lemma 6.1.2. For each vertex v of T, we define $\lambda(v)$ to be $\nu_{\mu(v)}(\boldsymbol{x})$, where $\boldsymbol{x}$ is the coordinate of v in $\triangle^{r-1}$. This provides a map $\lambda : V(\mathsf{T}) \to [r]$. Because of the assumption that the players are hungry, λ is a Sperner labeling, and Theorem 2.1.1 applies. There is thus a fully-labeled simplex.

Since the mesh size is taken arbitrarily small, the compactness of $\triangle^{r-1}$ and the closedness of the preference sets imply that there is a point $\boldsymbol{x}^* \in \triangle^{r-1}$ such that the $\nu_i(\boldsymbol{x}^*)$'s are all distinct. This point $\boldsymbol{x}^*$ is the sought envy-free division. □

6.1.1 Algorithmic features

The proof of Sperner's lemma (Theorem 2.1.1) can be made completely algorithmic. The algorithm underlying such a proof of Sperner's lemma can be used to compute, for any $\varepsilon > 0$, an "approximate" envy-free division.`To be completed`.

6.2 Necklace splitting

Two thieves have stolen a beautiful necklace with n beads on a string made of gold. Each bead is of a certain type, the total number of types being t. The thieves wish to divide the necklace, in such a way that each of them gets the same number of beads of each type. Assuming the number of beads per type being even, such a division clearly exists, at least by cutting the necklace between each pair of adjacent beads. We will see that a division with this notion of fairness not requiring more that t cuts exists, whatever the total number n of beads is. The following theorem was first proved by Goldberg and West Goldberg and West (1985) in 1985. A simpler proof using the Borsuk-Ulam theorem was proposed by Alon and West Alon and West (1986) the following year.

Denote by a_j the number of beads of type j. A division of the necklace is *fair* if for any bead type j, each thief gets at least $\lfloor \frac{a_j}{2} \rfloor$ beads of type j.

Theorem 6.2.1 (Necklace theorem)**.** *It is possible to split fairly the necklace between the two thieves with no more than t cuts.*

We give here the combinatorial proof proposed by Pálvolgyi. A sequence of elements in $\{+,-,0\}$ is *alternating* if there is no term equal to 0 and two consecutive terms are opposite. For $\boldsymbol{x} \in \{+,-,0\}^n$, we denote by $\operatorname{alt}(\boldsymbol{x})$ the longest alternating subsequence of $x_1,\ldots,x_n$.

Proof of Theorem 6.2.1. We arbitrarily assign to one of the thieves the sign $+$ and to the other the sign $-$. For a contradiction, we assume that all fair splittings require at least $t+1$ cuts. Let $g(\boldsymbol{x}) = \max\{\operatorname{alt}(\boldsymbol{y}) : \ \boldsymbol{y} \in \{+,-,0\}^n, \ \boldsymbol{y} \succcurlyeq \boldsymbol{x}\}$. We define a map $\lambda : \{+,-,0\}^n \setminus \{(0,\ldots,0)\} \rightarrow \{-1,+1,\ldots,-(n-1),+(n-1)\}$ as follows. Let $\boldsymbol{x} \in \{+,-,0\}^n \setminus \{(0,\ldots,0)\}$.

If $g(\boldsymbol{x}) \geqslant t+1$, we define $\lambda(\boldsymbol{x})$ to be $\varepsilon g(\boldsymbol{x})$, where $\varepsilon = +$ if the $\boldsymbol{y}$ for which the maximum is attained in the definition of g starts with a $+$, and $\varepsilon = -$ otherwise. There is no ambiguity in the definition: there may be several $\boldsymbol{y}$ attaining the maximum, but they all start with the same sign.

If $g(\boldsymbol{x}) \leqslant t$, we define $\lambda(\boldsymbol{x})$ as follows. For each i, we give the bead situated at position i to the $+$ thief if $x_i = +$, to the $-$ thief if $x_i = -$, and to none of the thieves if $x_i = 0$. Suppose that for each type j, each thief gets at most $a_j/2$ beads of this type. We can complete $\boldsymbol{x}$ in order to give to each thief at least $a_j/2$ beads of each type j. By definition of $g(\boldsymbol{x})$, it is a fair splitting with at most t cuts. Since we have supposed that such a splitting does not exist, it implies that there is at least one type j^* such that one of the thief gets strictly more than $a_{j^*}/2$ beads of this type. Choose such a j^*

as small as possible, and define ε to be the sign of the thief getting strictly more that $a_{j^*}/2$ beads of type j^*. We define then $\lambda(\boldsymbol{x})$ to be εj^*.

It can be checked that λ satisfies the condition of Theorem 2.4.2, which is impossible since n is larger than $n-1$. Thus the starting assumption cannot be done and there is a splitting as wished. □

6.2.1 Algorithmic features

6.2.2 Generalizations

CHAPTER 7

d-intervals

7.1 Statement

A *d-interval* is the union of at most d intervals of the real line. Given a family $\mathcal{H}$ of d-intervals, we denote by $\nu(\mathcal{H})$ the maximal number of pairwise disjoint d-intervals and by $\tau(\mathcal{H})$ the minimal size of a subset of points meeting each element of $\mathcal{H}$. Tardos for the case $d = 2$ and Kaiser for the general case proved the following theorem.

Theorem 7.1.1.

$$\tau(\mathcal{H}) \leqslant (d^2 - d + 1)\nu(\mathcal{H}).$$

7.2 Proof

The proof we give here is due to Aharoni, Kaiser, and Zerbib, and uses Shapley's theorem stated and proved in Chapter 2. It also uses Füredi's theorem (Theorem 1.5.1).

Proof of Theorem 7.1.1. Since $\mathcal{H}$ is finite, we can assume that all d-intervals are contained in $(0, 1)$. Moreover, we can clearly suppose that all intervals are open. Suppose that $\tau(\mathcal{H}) > k$ for some nonnegative integer k. We are going to show that $\nu(\mathcal{H}) > \frac{k}{d^2-d+1}$.

To each point in $\boldsymbol{x} \in \triangle^k$, we associate the set $S(\boldsymbol{x}) = \{x_1, x_1+x_2, \ldots, x_1+x_2+\cdots+x_k\}$ of at most k points in $(0, 1)$. We define then a labeling λ on $\triangle^k$ with nonempty subsets of $[k + 1]$ as follows. Let $\boldsymbol{x} \in \triangle^k$. Since $\tau(\mathcal{H}) > k$, there is at least one d-interval in $\mathcal{H}$ that has an empty intersection with $S(\boldsymbol{x})$. Pick such a d-interval H. We define $\lambda(\boldsymbol{x})$ to be the exact set of indices i such that $(x_1 + \cdots + x_{i-1}, x_1 + \cdots + x_{i-1} + x_i)$ contains at least one interval from H. Note that $\lambda(\boldsymbol{x})$ is of size at most d.

Consider a triangulation T of $\triangle^k$ of arbitrarily small mesh size. The labeling λ satisfies the condition of Theorem 2.3.1. Indeed, if v is on the face F where the i-th coordinate is 0, the integer i will never be present in

$\lambda(v)$. Thus, the conclusion of the lemma holds: there is $\sigma \in \mathsf{T}$ such that the collection $\mathcal{F} = \lambda(V(\sigma))$ is balanced, which means that there is a weight function $w\colon \mathcal{F} \to \mathbb{R}_+$ such that

$$\sum_{f \in \mathcal{F}\colon i \in f} w(f) = 1 \qquad \text{for all } i \in [k+1].$$

Since each element in $\mathcal{F}$ is of size at most d, we have $\nu^*(\mathcal{F}) \geqslant \frac{k+1}{d}$. By Theorem 1.5.1, we get

$$\nu(\mathcal{F}) > \frac{k}{d^2 - d + 1}.$$

Hence, there is a matching M in $\mathcal{F}$ of size larger than $\frac{k}{d^2-d+1}$. Each edge in M is obtained with a distinct point $\boldsymbol{x}$ in $\triangle^k$, since the edges of $\mathcal{F}$ are the $\lambda(v)$ when v runs over the vertices of σ. The mesh of T was chosen arbitrarily: make it go to zero. We conclude by compactness and using the fact that the intervals are all open: the edges in M provide at the limit pairwise disjoint d-intervals. $\square$

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