Master 2 - Maths - Info
ENS Lyon
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Topological Combinatorics
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Final Exam

Duration 2h, *No electronic device can be used. Any written document is allowed.*

1. A multilabeled Sperner lemma

Let $\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n$ be the standard unit vectors. Given a triangulation T of the standard $(n-1)$-dimensional simplex $\triangle^{n-1}=\mathrm{conv}(\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n)$, a *Sperner labeling* is a map $\lambda\colon V(\mathsf{T})\to[n]$ such that

$$\lambda(v)=j \quad\Longrightarrow\quad \boldsymbol{e}_j \text{ is a vertex of the minimal face of } \triangle^{n-1} \text{ containing } v.$$

The usual Sperner lemma, seen during the course, states (in its minimal version) that, in a triangulation with a Sperner labeling, there always exists at least one simplex with all labels. The purpose of this exercise is to prove the following generalization of this result.

Theorem 1. *Let T be a triangulation of the standard $(n-1)$-dimensional simplex $\triangle^{n-1}$ and let $\lambda_1,\ldots,\lambda_m$ be m Sperner labelings. For any choice of positive integers $\ell_1,\ldots,\ell_n$ such that $\ell_1+\cdots+\ell_n=m+n-1$, there exists a simplex τ of T on which, for each j, the label j is used in at least ℓ_j labelings.*

The usual Sperner lemma is the case $m=1$ and $\ell_1=\cdots=\ell_n=1$. The Figure 1 illustrates the case $m=2$, $n=3$, $\ell_1=\ell_3=1$, $\ell_2=2$.

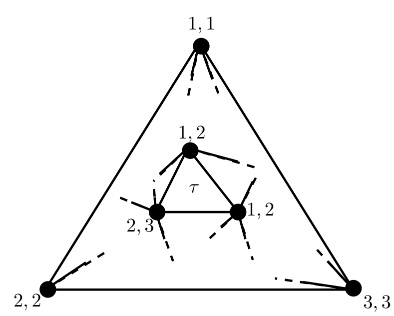


Figure 1. Illustration of Theorem 1.

Question 1. *Give an elementary proof of the case $n=2$.*

We identify a Sperner labeling λ_i with its *affine extension*: for a point $\boldsymbol{x}$ in $\triangle^{n-1}$, we define

$$\lambda_i(\boldsymbol{x})=\sum_{k=1}^{n}\alpha_k\boldsymbol{e}_{\lambda_i(v_k)},$$

where the v_k are the vertices of an $(n-1)$-dimensional simplex of T containing $\boldsymbol{x}$ and where the α_k are the weights of the v_k when one writes $\boldsymbol{x}$ as a convex combination of them. It is known that such an affine extension is well-defined and continuous.

We define $\overline{\lambda}\colon \triangle^{n-1} \to \mathbb{R}^n$ by

$$\overline{\lambda}(\boldsymbol{x}) = \frac{1}{m}\sum_{i=1}^{m} \lambda_i(\boldsymbol{x}).$$

Let $\boldsymbol{\beta} \in \mathbb{R}^n$ be defined by

$$\beta_j = \frac{\ell_j - 1}{m} + \frac{1}{mn}.$$

Question 2. *Prove that there exists an $\boldsymbol{x}^*$ such that $\overline{\lambda}(\boldsymbol{x}^*) = \boldsymbol{\beta}$.*

We introduce a bipartite graph $G = (V, E)$, whose vertices are partitioned into two sets I and J (the first of cardinality m and the second of cardinality n), and with an edge between $i \in I$ and $j \in J$ if and only if the jth coordinate of $\lambda_i(\boldsymbol{x}^*)$ is nonzero.

Question 3. *Prove that the degree of any vertex $j \in J$ is at least ℓ_j.*

Question 4. *Show how to finish then the proof of Theorem 1.*

2. Dol'nikov's inequality

A *hypergraph* is a set system $\mathcal{F}$ on a ground set X; the elements of X are the *vertices*, and the elements of $\mathcal{F}$ are the *hyperedges*. Many graph-theoretic notions can be extended to hypergraphs. A *proper k-coloring* of a hypergraph $\mathcal{F}$ on vertex set X is a function $c\colon X \to [k]$ such that no hyperedge $F \in \mathcal{F}$ is monochromatic. A graph is *k-colorable* if it admits a proper k-coloring.

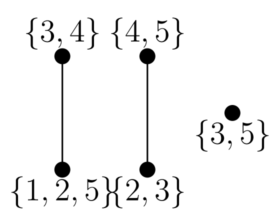
Given a hypergraph $\mathcal{F}$, define the 2-*colorability defect* $\operatorname{cd}(\mathcal{F})$ as the minimum number of vertices, such that removing them and all incident hyperedges from $\mathcal{F}$ results in a 2-colorable hypergraph. For instance, the hypergraph

$$\mathcal{F} = \big\{\{1,2,5\},\{2,3\},\{3,4\},\{3,5\},\{4,5\}\big\}$$

satisfies $\operatorname{cd}(\mathcal{F}) = 1$, because $\mathcal{F}$ is not 2-colorable, but deleting the vertex 5 and its incident edges $\{1,2,5\}$, $\{3,5\}$ and $\{4,5\}$ results in a 2-colorable hypergraph.

Question 5. *Show that if $\mathcal{F} = \binom{[n]}{k}$ (i.e., if $\mathcal{F}$ consists of all the k-element subsets of $[n]$), then* $\operatorname{cd}(\mathcal{F}) = n - 2k + 2$.

Given a hypergraph $\mathcal{F}$, the graph $\operatorname{KG}(\mathcal{F})$ is defined as the graph whose vertex set is $\mathcal{F}$ and whose edges are pairs of disjoint vertices. For instance, if $\mathcal{F}$ is the above hypergraph, then $\operatorname{KG}(\mathcal{F})$ is the graph below:



Note that if $\mathcal{F}$ consists of all the k-element subsets of an n-element set, then $\operatorname{KG}(\mathcal{F})$ is nothing but the usual *Kneser graph* $\operatorname{KG}(n,k)$. The aim of this exercise is to prove that $\chi(\operatorname{KG}(\mathcal{F})) \geqslant \operatorname{cd}(\mathcal{F})$. By Question 5, this generalizes the Lovász-Kneser theorem, which states that $\chi(\operatorname{KG}(n,k)) = n - 2k + 2$.

Let $d = \chi(\mathrm{KG}(\mathcal{F}))$, and fix a proper d-coloring of $\mathrm{KG}(\mathcal{F})$. As in the proof of the Lovász-Kneser theorem, we identify the ground set of $\mathcal{F}$ with a point set $X \subset \mathcal{S}^d$ in general position (no $d+1$ points lie on a hyperplane through the origin). For $\boldsymbol{x} \in \mathcal{S}^d$, we define $\boldsymbol{x} \in A_i$ if the open hemisphere $H(\boldsymbol{x})$ centered at $\boldsymbol{x}$ contains a set $F \in \mathcal{F}$ colored by color i, $i \in [d]$, and set $A_{d+1} = \mathcal{S}^d \setminus (A_1 \cup \cdots \cup A_d)$.

Question 6. *By applying an appropriate version of the Borsuk-Ulam theorem, show that there exists a point $\boldsymbol{y} \in S^d$ such that $\boldsymbol{y}, -\boldsymbol{y} \in A_i$ for some i.*

Question 7. *Show that $i = d+1$.*

Question 8. *Let $\mathcal{F}' = \{F \in \mathcal{F} \colon F \subset H(\boldsymbol{y}) \cup H(-\boldsymbol{y})\}$. Show that $\mathcal{F}'$ is 2-colorable.*

Question 9. *Deduce that $\mathrm{cd}(\mathcal{F}) \leqslant d$.*

Question 10. *Show that every simple graph is isomorphic to $\mathrm{KG}(\mathcal{F})$, for some suitable set system $\mathcal{F}$. (Hint: Given a graph G, let $S_1, S_2, \ldots, S_\ell$ be the inclusion-wise maximal stable sets of G – also known as independent sets. Then $uv \notin E(G)$ if and only if there exists $i \in [\ell]$ such that $u, v \in S_i$.)*

3. Tight bound on the connectivity of the stable set complex

3.1. **A lower bound.** Consider the following function ψ defined inductively for all simple graphs $G = (V, E)$ as follows.

$$\psi(G) = \begin{cases} 0 & \text{if } V = \varnothing \\ +\infty & \text{if } V \neq \varnothing \text{ and } E = \varnothing \\ \max\limits_{e \in E}\{\min\{\psi(G - e), \psi(G \divideontimes e) + 1\}\} & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} G - e &= (V, E \setminus \{e\}) \\ G \divideontimes e &= (V \setminus \{u, v, N(u), N(v)\}, E \setminus \{\delta(u), \delta(v)\}) \quad \text{for } e = uv. \end{aligned}$$

We denote by $\mathsf{S}(G)$ the stable set complex of a graph G and by $\mathrm{conn}(\mathsf{K})$ the connectivity of a simplicial complex K.

Question 11. *Prove that $\mathrm{conn}(\mathsf{S}(G)) \geqslant \psi(G) - 2$ for every simple graph G.*

It is known that this inequality is not tight: there are graphs for which it is strict. The purpose of this problem is to show that it is however tight when $\psi(G) \in \{0, 1\}$.

3.2. **Preliminary results.** The following two results will be useful for Section 3.3.

We denote by $\mathsf{K}_{(1)}$ the 1-*skeleton* of a simplicial complex K, formed by all simplices of K of dimension at most 1. It can be seen as a simple graph.

The following intuitive lemma can be taken for granted.

Lemma. *For a simplicial complex K, the inequality $\mathrm{conn}(\mathsf{K}) \geqslant 0$ holds if and only if the graph $\mathsf{K}_{(1)}$ is connected in the graph sense.*

Question 12. *Consider a graph H for which every pair u, v of non-adjacent vertices satisfies $N(u) \cap N(v) = \varnothing$. Prove that every connected component of H is a clique.*

(A *connected component* of a graph is an inclusion-wise maximal induced subgraph that is connected in the graph sense.)

3.3. **Case of tightness.** Recall that $\overline{G}$ is the complement of G, which is the graph with same vertex set and in which two vertices are adjacent if and only if they are not adjacent in G.

Question 13. *Prove that* $\psi(G) = 0$ *if and only if* $V = \varnothing$.

Question 14. *Suppose that* $\psi(G) = 1$ *and that* $G \divideontimes e$ *has no vertices for every edge* $e \in E$. *Prove that* $\overline{G}$ *is not connected (in the graph sense).*

Question 15. *Prove that actually* $\psi(G) = 1$ *always implies that* $\overline{G}$ *is not connected.*

Question 16. *Conclude that if* $\psi(G) \in \{0, 1\}$, *then* $\operatorname{conn}(\mathsf{S}(G)) = \psi(G) - 2$.

Question 17. *Prove that if* $\operatorname{conn}(\mathsf{S}(G)) \leqslant 0$, *then* $\operatorname{conn}(\mathsf{S}(G)) = \psi(G) - 2$ *as well.*