Exercises: Sperner's lemma and cake division

M2 Informatique Fondamentale : Topological Combinatorics

1. Spherical Sperner lemma

Consider a triangulation T of a d-dimensional sphere $\mathcal{S}^d$ and any labeling $\lambda\colon V(\mathsf{T}) \to [d+1]$.

Question. *Prove that there is an even number of fully-labeled simplices.*

Question. *Show how to derive from the previous result the usual Sperner lemma as stated in the course.*

2. Sperner's lemma for the prism and multi-cake division

Denote

$$\triangle_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ \colon x_1 + x_2 + x_3 = 1\} \qquad \text{and} \qquad \triangle_2 = \{(y_1, y_2) \in \mathbb{R}^2_+ \colon y_1 + y_2 = 1\}.$$

2.1. **Sperner's lemma for the prism.** Let $P = \triangle_3 \times \triangle_2$ and T a triangulation of P.

Consider $\lambda\colon V(\mathsf{T}) \to [3]\times[2]$ such that for $v \in V(\mathsf{T})$ with coordinates $\big((x_1, x_2, x_3), (y_1, y_2)\big)$, we have

$$\lambda(v) = (i, j) \Longrightarrow (x_i > 0 \text{ and } y_j > 0).$$

Question. *Prove the existence of a tetrahedron $\sigma \in \mathsf{T}$ such that $\lambda(V(\sigma))$ is of the form $\{(1,1),(2,1),(3,1),(t,2)\}$.*

2.2. **Two-player two-cake division.** We are given two cakes, each of them identified with a copy of the interval $[0,1]$, and two players, Alice and Bob, with preference functions. Each will get a piece of each cake. The issue here is that each preference function is not obtained by a preference function for the first cake and a preference function for the second cake: what Alice gets from the first cake influences what she wants from the second cake, and the same for Bob.

In this exercise, we decide to divide the first cake into three connected pieces and the second one into two connected pieces, and to assign a piece from each cake to each player. Note that doing this, a piece of the first cake will remain (see the last question of the exercise).

Identify such a division with a point in P, the point (x_1, x_2, x_3) in $\triangle_3$ encoding the lengths of the pieces from left to right in the first cake and the point (y_1, y_2) in $\triangle_2$ encoding the lengths of the pieces from left to right in the second cake. The preference function p_A of Alice (resp. p_B of Bob) maps a division $(\boldsymbol{x}, \boldsymbol{y}) \in P$ to the set of pairs $(i, j) \in [3] \times [2]$ that she (resp. he) prefers in that division. Formally, for $Z \in \{A, B\}$, we have a map $p_Z(\boldsymbol{x}, \boldsymbol{y}) \in 2^{[3]\times[2]} \setminus \{\varnothing\}$, with the property that if (i, j) is in $p_Z(\boldsymbol{x}, \boldsymbol{y})$, then $x_i > 0$ and $y_j > 0$. We have moreover a continuity assumption, as in the course.

Let $\varepsilon > 0$ and let T be a triangulation of P such that $\operatorname{diam}(\sigma) < \varepsilon$ for all $\sigma \in \mathsf{T}$. Consider $o\colon V(\mathsf{T}) \to \{A, B\}$ such that any tetrahedron σ in T has exactly two A's and two B's (i.e., $|o^{-1}(A) \cap V(\sigma)| = |o^{-1}(B) \cap V(\sigma)| = 2$).

Question. *For a specific* T*, such a labeling* o *may not exist. Show that there always exists a triangulation for which such a labeling does exist, and that the diameter of the triangulation can be chosen arbitrary small.*

Question. *With the help of the result of Section 2.1, show there exists* $(\boldsymbol{x}, \boldsymbol{y}) \in P$ *and* $t \in [3]$ *such that* $p_A(\boldsymbol{x}, \boldsymbol{y})$ *contains two elements from* $\{(1,1),(2,1),(3,1),(t,2)\}$ *and* $p_B(\boldsymbol{x}, \boldsymbol{y})$ *contains the two other elements.*

Question. *Prove that there exists an envy-free division of the two cakes.*

Question. *Build an example showing that in general, it is not possible to achieve an envy-free division in this setting with two connected pieces in each cake.*

3. Discrete envy-free cutting

Consider an open necklace with n types of beads, which has to be divided between n players. The beads are numbered from 1 to n and so are the players. The division has to be done in such a way that player i gets at least as many beads of type i as any other player. The beads cannot be cut.

Question. *Prove that this can be achieved with* $n-1$ *cuts.*

4. Sperner's lemma for trees

The objective of this exercise consists in proving the following theorem.

Theorem 1. *Let* $T = (V, E)$ *be a tree with at least two vertices and* L *a subset of* V*. Consider a map* $\lambda\colon V \to 2^L$*. Suppose that for every* $v \in V$*, the elements in* $L \setminus \lambda(v)$ *belong to the vertex set of some connected component of* $T - v$*. Then there is an edge* uv *with* $\lambda(u) \cup \lambda(v) = L$*.*

From now on, we consider a tree and a map as in the statement of the theorem.

Question. *Prove that we have* $v \in \lambda(v)$ *for every* $v \in L$*.*

Denote by K_v a connected component of $T - v$ containing the vertices in $L \setminus \lambda(v)$. We build a sequence of vertices as follows. Let v_0 be any vertex. We define then v_n to be the neighbor of v_{n-1} in $K_{v_{n-1}}$.

Question. *Prove that if there is no vertex* v *such that* $\lambda(v) = L$*, then the sequence* (v_n) *is periodic of period* 2*, for* n *large enough.*

Question. *Show how to conclude the proof of the theorem.*

Application: a fixed-point theorem for trees.

Theorem 2. *Let* $T = (V, E)$ *be a tree and* f *any self-map of* V*. Suppose* f *has no fixed-point. Then there is an edge* $uv \in E$ *such that* uv *belongs to the unique* u*-*v *path of* T*.*

Question. *Show how Theorem 1 can be used to prove Theorem 2.*

Application: Helly's theorem for trees.

Theorem 3. *Let* $T_1, \ldots, T_s$ *be subtrees of a tree* $T = (V, E)$*. If* $V(T_i) \cap V(T_j) \neq \varnothing$ *for every* $i \neq j$*, then* $\bigcap_{i=1}^{s} V(T_i) \neq \varnothing$*.*

Question. *Show how Theorem 1 can be used to prove Theorem 3.*