Exercises: Surjectivity lemma and envy-free divisions

M2 Informatique Fondamentale : Topological Combinatorics

1. Alternate proof of the surjectivity lemma for the standard simplex

Consider a continuous self-map f of the standard simplex such that $f(F) \subseteq F$ for every face F. According to the surjectivity lemma, such a map is then necessarily surjective.

Question. *Show how this fact can be proved from Sperner's lemma.*

2. Even more general?

Question. *Is it true that any continuous self-map f of B^d mapping its boundary to itself is surjective?*

3. Cake-cutting with a vanishing player

We consider the cake-cutting setting, as in the course. We saw that when there are $n-1$ players, there always exists a division of the cake into n connected pieces so that whatever the piece is taken by an extra player, there is an envy-free assignment of the remaining pieces to the $n-1$ players.

Actually, a dual statement holds: when there are n players, there always exists a division of the cake into $n-1$ connected pieces so that whoever the player vanishes, there is still an envy-free assignment of the $n-1$ pieces to the remaining players.

Question. *Prove this dual statement.*

4. Rental harmony

Roommates, numbered from 1 to n, want to share an apartment with n rooms, also numbered from 1 to n. They have to decide how to assign each of them to a room and how to divide the rent between them. The total rent R is fixed and thus the question is about the proportion, encoded as a vector $(x_1, \ldots, x_n) \in \triangle^{n-1}$: the rent of room 1 is $x_1 R$, the rent of room 2 is $x_2 R$, and so on.

Each player i has a *choice function* $c_i \colon \triangle^{n-1} \to 2^{[n]} \setminus \{\varnothing\}$: given a rent division $\boldsymbol{x} \in \triangle^{n-1}$, the rooms with which player i will be happy are those in $c_i(\boldsymbol{x})$. Two natural assumptions are made:

- if room j belongs to $c_i(\boldsymbol{x}^{(k)})$ for a converging sequence of $\boldsymbol{x}^{(k)}$'s, then j belongs to $c_i(\boldsymbol{x}^{(\infty)})$, where $\boldsymbol{x}^{(\infty)} = \lim_{k\to+\infty} \boldsymbol{x}^{(k)}$.
- if room j is such that $x_j = 0$, then $j \in c_i(\boldsymbol{x})$. (If a room is free, any player is happy to be assigned this room.)

A rent division $\boldsymbol{x}$ is *envy-free* if there is a bijective assignment π of the players to the rooms (player i gets room $\pi(i)$) such that $\pi(i) \in c_i(\boldsymbol{x})$ for every player i. This exercise aims at proving that when the above two assumptions are satisfied, then there always exists an envy-free rent division.

We start as for the cake-cutting with a triangulation T of $\triangle^{n-1}$ of arbitrary small mesh size. We also assume that there is an owner labeling $o\colon V(\mathsf{T}) \to [n]$ such that every $(n-1)$-dimensional simplex of T is fully-labeled with respect to o.

Question. *Show how to define a labeling* $\lambda\colon V(\mathsf{T}) \to 2^{[n]} \setminus \{\varnothing, [n]\}$ *such that* $\lambda(v) \subseteq c_{o(v)}(v)$.

Question. *Show how to define a map* $g\colon 2^{[n]} \setminus \{\varnothing, [n]\} \to [n]$ *such that* $g(X) \notin X$ *and* $g(X) - 1 \in X$ *for all* $X \in 2^{[n]} \setminus \{\varnothing, [n]\}$. *(Here* $1 - 1 = n$*.)*

The map $f\colon v \in V(\mathsf{T}) \to \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ defined by $f(v) = \boldsymbol{e}_{g(\lambda(v))}$ maps each vertex of T to a vertex of $\triangle^{n-1}$. Consider the affine extension of f on every simplex of T, which we still denote by f. Observe that f is a self-map of $\triangle^{n-1}$.

Question. *Use this observation to finish the proof of the existence of the envy-free rental division.*

5. 1-intervals

We have claimed during the course that for any family $\mathcal{F}$ of closed intervals of $\mathbb{R}$, we have $\tau(\mathcal{F}) = \nu(\mathcal{F})$.

Proof. Provide an elementary proof of this statement. □