Exercises: Stable set complex, Meshulam’s lemma, and related topics

M2 Informatique fondamentale : Topological Combinatorics

1. Strengthening of Haxell’s theorem

Haxell proved the following statement: let \( G \) be a graph whose vertices are partitioned into subsets \( V_1, \ldots, V_m \); if each \( V_i \) has cardinality at least \( 2\Delta(G) \), then there exists a stable set \( S \) such that \( |S \cap V_i| = 1 \) for all \( i \).

Question. Show that the proof provided in the course can be slightly adapted to get that, under the same condition, given any vertex \( v \) of \( G \), we can additionally request that \( v \in S \).

2. Du-Hsu-Wang conjecture

We denote by \( I(G) \) the stable set complex of a graph \( G \).
Let \( P_n \) (resp. \( C_n \)) be the path (resp. the circuit) with \( n \) vertices (seen as graphs).

Question. Prove that \( \text{conn}(I(P_n)) \geq \frac{n}{3} - 2 \). (Indication: try induction.)

Question. Deduce from the previous question that \( \text{conn}(I(C_n)) \geq \frac{n}{3} - 2 \) as well.

The following theorem of Fleischner and Stiebitz was originally conjectured by Du, Hsu, and Wang.

Theorem 1. Let \( k \) be a positive integer. Suppose that the vertices of \( C_{3k} \) are partitioned into subsets \( V_1, \ldots, V_k \), all of cardinality equal to three. Then there exists a stable set \( S \) such that \( |S \cap V_i| = 1 \) for all \( i \).

Question. Use the result of the previous questions to prove Theorem 1.

The next question proposes an alternate proof of Theorem 1. Actually, it proves an even stronger result.

The Schrijver graph \( SG(3k, k) \) is the graph whose vertices are the stable sets of \( C_{3k} \) of cardinality \( k \), and whose edges connect vertices whose corresponding stable sets are disjoint. A theorem of Schrijver ensures that \( \chi(SG(3k, k)) = k + 2 \).

Question. Use Schrijver’s theorem to get another proof of Theorem 1.

Question. What is actually the stronger result obtained by this latter proof?

3. Hypergraph matchings

The line-graph \( L(H) \) of a hypergraph \( H \) is the graph whose vertices are the edges of \( H \) and in which vertices are adjacent when the corresponding edges in \( H \) intersect.
Suppose now that \( H \) is an \( r \)-uniform hypergraph (all edges are of cardinality \( r \)).
Question. Prove that for any subgraph $G$ of $L(\mathcal{H})$ and any matching $M$ of $\mathcal{H}$ using only vertices of $G$, we have

$$\text{conn}(I(G)) \geq \frac{|M|}{r} - 2.$$  

(Indication: use $\gamma_I$.)

A theorem of Aharoni and Haxell goes as follows.

**Theorem 2.** Let $\mathcal{H}_i = (V, E_i)$ for $i = 1, \ldots, m$ be $r$-uniform hypergraphs on a same vertex set $V$. Suppose that for every $I \subset [m]$ there is a matching of size at least $r(|I| - 1) + 1$ in the hypergraph $(V, \bigcup_{i \in I} E_i)$. Then there exists a matching of $(V, \bigcup_{i=1}^{m} E_i)$ with an edge from each $\mathcal{H}_i$.

**Question.** Show how to prove Theorem 2 from the previous question.

**Question.** Why can Theorem 2 be considered as a generalization of Hall’s marriage theorem?