Exercises: Stable set complex, Meshulam's lemma, and related topics

M2 Informatique Fondamentale : Topological Combinatorics

1. Strengthening of Haxell's theorem

Haxell proved the following statement: let G be a graph whose vertices are partitioned into subsets $V_1, \ldots, V_m$; if each V_i has cardinality at least $2\Delta(G)$, then there exists a stable set S such that $|S \cap V_i| = 1$ for all i.

Question. *Show that the proof provided in the course can be slightly adapted to get that, under the same condition, given any vertex v of G, we can additionally request that $v \in S$.*

2. Du-Hsu-Wang conjecture

We denote by $\mathsf{I}(G)$ the stable set complex of a graph G.

Let P_n (resp. C_n) be the path (resp. the circuit) with n vertices (seen as graphs).

Question. *Prove that* $\operatorname{conn}(\mathsf{I}(P_n)) \geqslant \frac{n}{3} - 2$. *(Indication: try induction.)*

Question. *Deduce from the previous question that* $\operatorname{conn}(\mathsf{I}(C_n)) \geqslant \frac{n}{3} - 2$ *as well.*

The following theorem of Fleischner and Stiebitz was originally conjectured by Du, Hsu, and Wang.

Theorem 1. *Let k be a positive integer. Suppose that the vertices of C_{3k} are partitioned into subsets $V_1, \ldots, V_k$, all of cardinality equal to three. Then there exists a stable set S such that $|S \cap V_i| = 1$ for all i.*

Question. *Use the result of the previous questions to prove Theorem 1.*

The next question proposes an alternate proof of Theorem 1. Actually, it proves an even stronger result.

The Schrijver graph $\mathrm{SG}(3k, k)$ is the graph whose vertices are the stable sets of C_{3k} of cardinality k, and whose edges connect vertices whose corresponding stable sets are disjoint. A theorem of Schrijver ensures that $\chi(\mathrm{SG}(3k, k)) = k + 2$.

Question. *Use Schrijver's theorem to get another proof of Theorem 1.*

Question. *What is actually the stronger result obtained by this latter proof?*

3. Hypergraph matchings

The *line-graph* $L(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the graph whose vertices are the edges of $\mathcal{H}$ and in which vertices are adjacent when the corresponding edges in $\mathcal{H}$ intersect.

Suppose now that $\mathcal{H}$ is an r-uniform hypergraph (all edges are of cardinality r).

Question. *Prove that for any subgraph G of $L(\mathcal{H})$ and any matching M of $\mathcal{H}$ using only vertices of G, we have*

$$\operatorname{conn}(\mathsf{I}(G)) \geqslant \frac{|M|}{r} - 2.$$

(Indication: use γ^I.)

A theorem of Aharoni and Haxell goes as follows.

Theorem 2. *Let $\mathcal{H}_i = (V, E_i)$ for $i = 1, \ldots, m$ be r-uniform hypergraphs on a same vertex set V. Suppose that for every $I \subseteq [m]$ there is a matching of size at least $r(|I| - 1) + 1$ in the hypergraph $(V, \bigcup_{i \in I} E_i)$. Then there exists a matching of $(V, \bigcup_{i=1}^{m} E_i)$ with an edge from each $\mathcal{H}_i$.*

Question. *Show how to prove Theorem 2 from the previous question.*

Question. *Why can Theorem 2 be considered as a generalization of Hall's marriage theorem?*