Topological bounds for graph representations

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Plan

Graph representations

Orthogonal representation[1] of a graph $G = (V, E)$: assignment of a vector $\boldsymbol{x}_v \in \mathbb{R}^t$ to each vertex v such that

- $\boldsymbol{x}_v \neq \mathbf{0}$ for all $v \in V$.
- $\langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = 0$ for all $uv \in E$.

[1] This definition is often given for the complement of G instead.

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Orthogonality dimension of G: minimal t s.t. there exists an orthogonal representation. Denoted by $\xi_{\mathbb{R}}(G)$.

Figure: Two examples of orthogonal representations; both graphs have actually orthogonality dimension equal to 3.

[1] This definition is often given for the complement of G instead.

Relevance

Plays an important role in information theory, e.g., for the Shannon capacity $\Theta(G) = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)}$.
(Here the graph product is the strong product $\boxtimes$.)

Lovász (1979): $\Theta(G) \leqslant \xi_{\mathbb{R}}(\overline{G})$.

Also useful in the theory of geometric representations of graphs: e.g., Lovász, Saks, and Schrijver (1989) gave a geometric characterization of d-connectivity in graphs using orthogonal representations of dimension d.

Graph representations over any field

Orthogonal representation[2] of a graph $G = (V, E)$ over a field $\mathbb{F}$: assignment of a vector $\boldsymbol{x}_v \in \mathbb{F}^t$ to each vertex v such that

- $\langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle \neq 0$ for all $v \in V$.
- $\langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = 0$ for all $uv \in E$.

Orthogonality dimension of G: minimal t s.t. there exists an orthogonal representation. Denoted by $\xi_{\mathbb{F}}(G)$.

Figure: Two examples of orthogonal representations over $\mathbb{Z}_2$; both graphs have actually orthogonality dimension equal to 3.

[2]This definition is often given for the complement of G instead.

Relevance

Orthogonal representations over finite fields: introduced by Peeters (1996) as a tool to study some complexity problems for graph colorings.

Orthogonal representations over $\mathbb{C}$: introduced by de Wolf (2001) to study quantum communication (information theory)

Properties

We have for any field $\mathbb{F}$:

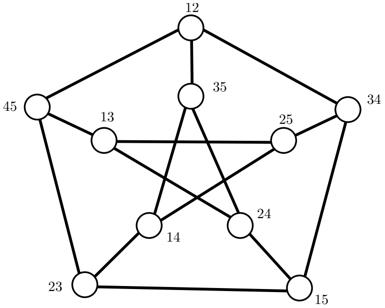
$$\omega(G) \leqslant \xi_{\mathbb{F}}(G) \leqslant \chi(G).$$

- First inequality: orthogonal representation of a clique provides independent vectors.
- Second inequality: unit vectors provide a graph representation.

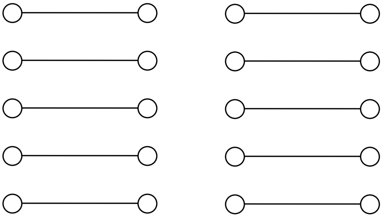
Kneser graphs

$KG_{n,k}$, with $n \geqslant 2k-1$, is the Kneser graph with

$$V = \binom{[n]}{k}$$
$$E = \left\{ XY \in \binom{V}{2} : X \cap Y = \varnothing \right\}$$



$KG_{5,2}$ is the Petersen graph



$KG_{6,3}$ is the matching M_{10}

"Historical importance" of Kneser graphs

Theorem (Lovász 1978)

$\chi(\mathsf{KG}_{n,k}) = n - 2k + 2.$

Conjectured by Kneser (1955).

$\chi(\mathsf{KG}_{n,k}) \leqslant n - 2k + 2$ noticed by Kneser.

$\chi(\mathsf{KG}_{n,k}) \geqslant n - 2k + 2$ requires use of algebraic topology and topological lower bounds.

1978: birth of topological combinatorics.

Relevance of Kneser graphs

Kneser graphs have many interesting properties, e.g.,

- triangle-free graphs with arbitrarily large chromatic number.
- graphs with arbitrarily large ratio between fractional and usual chromatic numbers:

$$\chi^f(\mathsf{KG}_{n,k}) = \frac{n}{k} \qquad \chi(\mathsf{KG}_{n,k}) = n - 2k + 2.$$

Kneser graphs and graph representations

Theorem (Haviv 2019)

- $\xi_{\mathbb{R}}(\mathsf{KG}_{n,k}) = n - 2k + 2$.
- $\xi_{\mathbb{C}}(\mathsf{KG}_{n,k}) \geqslant \frac{1}{2}(n - 2k + 2)$.

Since $\xi^{f}_{\mathbb{R}}(\mathsf{KG}_{n,k}) \leqslant \frac{n}{k}$: examples of arbitrarily large ratio between fractional orthogonality dimension and orthogonality dimension.

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Fractional orthogonality dimension: there is a systematic way for introducing fractional counterparts of graph parameters (Hu, Tamo, Shayevitz 2018).

Topological lower bounds on orthogonality dimension

A topological lower bound:

Theorem (Dol'nikov 1993)

For any graph G, we have $\chi(G) \geqslant \mathrm{cd}_2(\mathcal{K}_G)$.

Haviv showed that the topological method can also be applied for bounding the orthogonality dimension over $\mathbb{R}$ and $\mathbb{C}$:

Theorem (Haviv 2019)

For any graph G, we have

- $\xi_{\mathbb{R}}(G) \geqslant \mathrm{cd}_2(\mathcal{K}_G)$.
- $\xi_{\mathbb{C}}(G) \geqslant \frac{1}{2}\,\mathrm{cd}_2(\mathcal{K}_G)$.

All bounds and all fields

Actually many topological lower bounds, with a clear hierarchy:

$$\begin{array}{rlll} \chi(G) & \geqslant \mathrm{Xind}(\mathrm{H}(G))+2 & \geqslant & \mathrm{ind}(\mathrm{H}(G))+2 \\ & \geqslant \mathrm{ind}(\mathrm{B}_0(G))+1 & \geqslant & \mathrm{coind}(\mathrm{B}_0(G))+1 \\ & \geqslant \mathrm{coind}(\mathrm{H}(G))+2 & \geqslant & n-\mathrm{alt}(\mathcal{K}_G) \\ & \geqslant \mathrm{cd}_2(\mathcal{K}_G). & & \end{array}$$

(In the case of Kneser graphs $\mathrm{KG}_{n,k}$, all these bounds are equal to $n-2k+2$.)

Theorem (Alishahi, M. 2019+)

For any graph G and field $\mathbb{F}$, we have $\xi_{\mathbb{F}}(G) \geqslant \mathrm{Xind}(\mathrm{H}(G))+2$.

Corollary (Alishahi, M. 2019+)

$\xi_{\mathbb{F}}(\mathrm{KG}_{n,k}) = n-2k+2$ independently of the field $\mathbb{F}$.

Plan

1. Graph representations and Kneser graphs
2. Cross-index and Hom-complex
3. Generalization: matroids and rainbow balanced bicliques
4. Open questions

Cross-index

Z_2-poset: poset $P = (X, \preccurlyeq)$ with a fixed-point free order-preserving involution ν, i.e.:

- $\nu^2 = \text{id}$.
- $\nu(x) \neq x$ for all $x \in X$.
- $\nu(x) \preccurlyeq \nu(y)$ when $x \preccurlyeq y$.

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Cross-index of P:

- smallest t s.t. there is an order-preserving Z_2-map $P \to Q_t$.
- denoted by $\mathrm{Xind}(P)$.
- introduced by Simonyi, Tardif, and Zsbán (2013).

Hom-complex

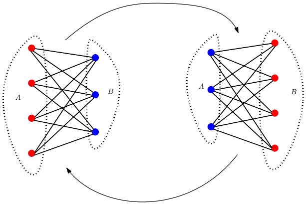
Hom-complex of a graph $G = (V, E)$:

$$\{(A, B)\colon A, B \in 2^V \setminus \{\varnothing\},\ A \cap B = \varnothing,\ G[A, B] \text{ is a bliclique }\}$$

with $(A, B) \preccurlyeq (A', B')$ if $A \subseteq A'$ and $B \subseteq B'$.

Denoted by $\mathrm{H}(G)$.

It is a Z_2-poset:



So, it has a cross-index $\mathrm{Xind}(\mathrm{H}(G))$.

Cross-index is a lower bound (for coloring)

Proof of $\chi(G) \geqslant \mathrm{Xind}(\mathrm{H}(G)) + 2$:

Consider a proper coloring $c\colon V \to [m]$.

Define $\phi(A, B) = \pm \min(c(A \cup B))$,

- with '$+$' if $c(A) < c(B)$.
- with '$-$' if $c(A) > c(B)$.

It is an order-preserving Z_2-map $\mathrm{H}(G) \to Q_{m-2}$.

$\mathrm{Xind}(\mathrm{H}(G))$ = smallest t s.t. there is an order-preserving Z_2-map $\mathrm{H}(G) \to Q_t$.

Thus $\chi(G) \geqslant \mathrm{Xind}(\mathrm{H}(G)) + 2$. □

Cross-index is a lower bound (for orthogonality dimension)

Proof of $\xi_{\mathbb{F}}(G) \geqslant \mathrm{Xind}(\mathrm{H}(G)) + 2$:

Consider an s-dimensional orthogonal representation $(\boldsymbol{x}_v)_{v \in V}$.

Choose any total order $\leqslant$ on the subspaces of $\mathbb{F}^s$.

Define $\phi(A, B) = \pm(\dim \mathcal{V}(A) + \dim \mathcal{V}(B) - 1)$,

- with '$+$' if $\mathcal{V}(A) < \mathcal{V}(B)$,
- with '$-$' if $\mathcal{V}(A) > \mathcal{V}(B)$,

where $\mathcal{V}(U) = \operatorname{span}\{\boldsymbol{x}_v \colon v \in U\}$.

Since $\mathcal{V}(A) \subseteq \mathcal{V}(B)^\perp$, we have $\dim \mathcal{V}(A) + \dim \mathcal{V}(B) \leqslant s$, and thus ϕ is an order-preserving Z_2-map $\mathrm{H}(G) \to Q_{s-2}$.

Thus $\xi_{\mathbb{F}}(G) \geqslant \mathrm{Xind}(\mathrm{H}(G)) + 2$. □

Plan

Independent representation over a matroid

A crucial property used in the previous proof:

- $\boldsymbol{x}_v \notin \mathcal{V}(N(v))$ for all $v \in V$.

Independent representation over a matroid M: assignment of an element $x_v \in M$ to each vertex v s.t. $x_v \notin \text{span}\{x_u : u \in N(v)\}$.

Common generalization of proper colorings and orthogonal representations:

- proper colorings are independent representations over the free matroid (all subsets are independent)
- orthogonal representations are independent representations over the linear matroid

Rainbow bicliques

The same proof as before gives:

Theorem

If G has an independent representation over a matroid M, then there is a biclique $G[A, B]$ s.t.

- the elements on each side form an independent set.
- $|A| + |B| = \mathrm{Xind}(\mathrm{H}(G)) + 2$.

Rainbow balanced bicliques

With the help of a “Ky Fan lemma” for the cross-index (Alishahi, Hajiabolhassan, M. 2017), we get the existence of a balanced biclique.

Theorem (Alishahi, M. 2019+)

If G has an independent representation over a matroid M, then there is a biclique $G[A, B]$ s.t.

- the elements on each side form an independent set.
- $|A| \approx |B| \approx \frac{1}{2}\,\mathrm{Xind}(\mathrm{H}(G)) + 1$.

The last item implies $\mathrm{rk}\, M \geqslant \frac{1}{2}\,\mathrm{Xind}(\mathrm{H}(G)) + 2$.

Some corollaries (1)

Local chromatic number of a graph G:

$$\chi^{\ell}(G) = \min \max_{v \in V} |c(N[v])|,$$

where the minimum is taken over all proper colorings c.

Corollary (Simonyi, Tardos 2006)

If $\chi(G) \geqslant t$ for "topological reasons", then $\chi^{\ell}(G) \geqslant t/2 + 1$.

Proof: use $M = U_m^r$ (uniform matroid of rank r with m elements).

Some corollaries (2)

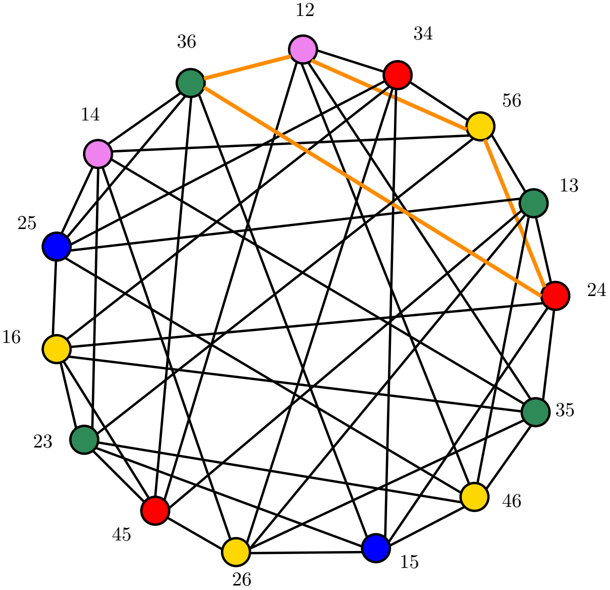
Corollary (Simonyi, Tardos 2006; Simonyi, Tardif, Zsbán 2013)

If $\chi(G) \geqslant t$ for "topological reasons", then there is a rainbow balanced biclique of size t in every proper coloring of G.

rainbow = the vertices get distinct colors
balanced = the sides are almost the same size

Proof: use the free matroid.

A rainbow biclique in $KG_{6,2}$



Some corollaries (3)

$V \times V$ matrix A represents a graph G over $\mathbb{F}$ if

- $A_{v,v} \neq 0$ for every vertex v.
- $A_{u,v} = 0$ for every adjacent vertices u, v.

Min-rank parameter[3] of a graph G over a field $\mathbb{F}$:

$$\mathrm{minrk}_{\mathbb{F}}(G) = \min\{\mathrm{rk}(A) \colon A \text{ represents } G \text{ over } \mathbb{F}\}.$$

(Due to Haemers (1981); provides good upper bounds on Shannon capacity.)

Corollary (Alishahi, M. 2019+)

$\mathrm{minrk}_{\mathbb{F}}(G) \geqslant \frac{1}{2}\,\mathrm{Xind}(\mathrm{H}(G)) + 2$ for any field $\mathbb{F}$.

(Haviv (2019) proved $\mathrm{minrk}_{\mathbb{R}}(G) \geqslant \sqrt{\frac{\mathrm{cd}_2(\mathcal{K}_G)}{2}}$.)

[3] Usually, this definition is given for the complement of G instead.

Fan lemma for the cross-index

Lemma (Alishahi, Hajiabolhassan, M. 2017)

Let $(P, \preccurlyeq)$ be a free Z_2-poset and $s = \text{Xind}(P)$. Consider an order-preserving Z_2-map $\phi\colon P \to Q_r$ with $r \geqslant s$. Then there exists a chain $p_1 \prec p_2 \prec \cdots \prec p_r$ such that

$$0 < -\phi(p_1) < +\phi(p_2) < -\phi(p_3) < \cdots < (-1)^r\phi(p_r).$$

Moreover, if $r = s$, then for every $\boldsymbol{\varepsilon} \in \{+,-\}^r$ there exists a chain $p_1 \prec p_2 \prec \cdots \prec p_r$ s.t.

$$0 < \varepsilon_1\phi(p_1) < \varepsilon_2\phi(p_2) < \varepsilon_3\phi(p_3) < \cdots < \varepsilon_r\phi(p_r).$$

This answers a question of Simonyi, Tardif, and Zsbán 2013.
It shows that when the coloring or the orthogonal representation is optimal, much more can be said about "rainbow" bicliques.

Plan

Better bounds for some matroids?

Let $t = \mathrm{Xind}(\mathrm{H}(G)) + 2$.

Theorem

Assume G has an independent representation over a matroid M. Then there is a biclique $G[A, B]$ s.t.

- the elements on each side form an independent set.
- $|A| \approx |B| \approx t/2 + 1$.

In particular $\mathrm{rk}(M) \geqslant t/2 + 2$.

When M is a linear matroid and we require orthogonality for adjacent vertices, then we get actually $\mathrm{rk}(M) \geqslant t$.
Same thing when M is the free matroid: proper colorings.

Question

Under which general condition do we have $\mathrm{rk}(M) \geqslant t$?

A “naive” condition

Under the condition:

$$\mathrm{rk}\{x_v : v \in A\} + \mathrm{rk}\{x_v : v \in B\} \leqslant \mathrm{rk}(M) \text{ when } G[A,B] \text{ is a balanced biclique,} \quad (\star)$$

we have the conclusion $\mathrm{rk}(M) \geqslant t$.

$(\star)$ is probably the most general condition and covers the previous cases.

But

Proposition (Alishahi, M. 2019+)

Deciding whether an independent representation satisfies the condition above is NP-complete.

The existence of a general polynomial condition covering the previous cases remains open.

Thank you.