A mutilabeled version of Fan's lemma

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Joint work with Francis E. Su.

The original Fan lemma

Sperner's lemma and its multilabeled generalization

The multilabeled version of Fan's lemma

Applications

- Rainbow bipartite subgraphs
- Splitting necklaces
- Multilabeled Sperner lemma

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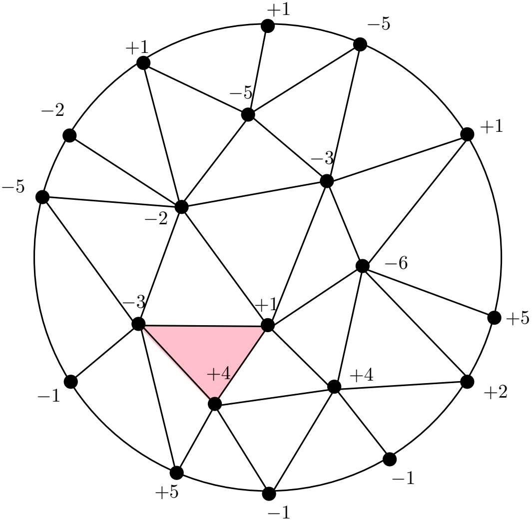
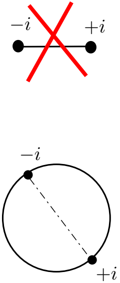
Labeling in dimension two

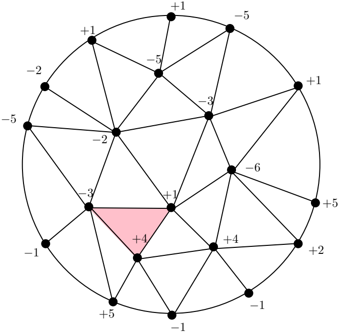


RULES

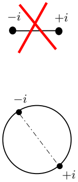


Fan's lemma

Ky Fan (1914-2010)



RULES



Fan's lemma

Theorem (Fan (1952))

Let T *be a centrally symmetric triangulation of* $\mathcal{S}^d$*. Consider a labeling* $\lambda\colon V(\mathsf{T}) \to \mathbb{Z} \setminus \{0\}$ *such that*

- $\lambda(u) + \lambda(v) \neq 0$ *for all pairs of adjacent vertices* u, v.

  (There is no complementary edge.)
- $\lambda(-v) = -\lambda(v)$ *for all* $v \in V(\mathsf{T})$.

  (λ is antipodal.)

$\Rightarrow$ *There is an alternating* d*-simplex.*

d-simplex σ is alternating if $\sigma = \langle v_0, \ldots, v_d \rangle$ with

$$0 < +\lambda(v_0) < -\lambda(v_1) < +\lambda(v_2) < \cdots < (-1)^d \lambda(v_d)$$

or with

$$0 < -\lambda(v_0) < +\lambda(v_1) < -\lambda(v_2) < \cdots < (-1)^{d+1} \lambda(v_d).$$

Fan's lemma: applications

Fan's lemma has many applications, especially in topological combinatorics:

- ♣ Easy and constructive proof of the Borsuk-Ulam theorem.
- ♣ Circular chromatic number of Kneser graphs, and variations (M. 2005, Simonyi-Tardos 2006, Chen 2011, Alishahi-Hajiabolhassan 2015, etc.).
- ♣ Local chromatic number (Simonyi-Tardos 2006).
- ♣ Quadrangulations of projective spaces (Kaiser-Stehlík 2015).
- ♣ Fair divisions (Simonyi 2008).
- ♣ Short proof of the Babson-Kozlov-Lovász theorem (on $\mathrm{Hom}(C_{2r+1}, G)$).

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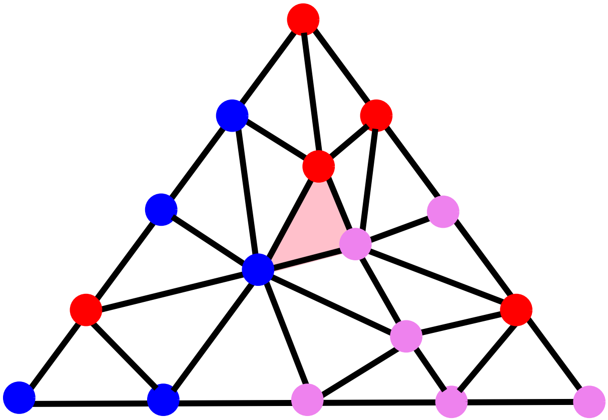
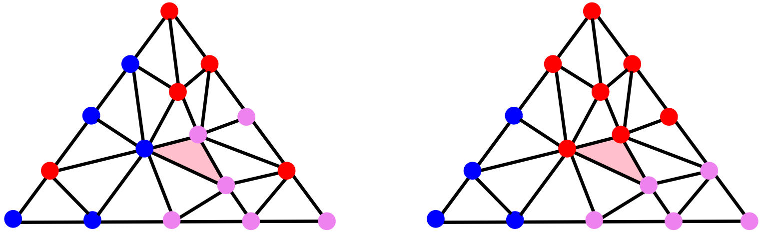
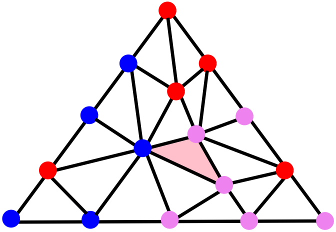
Sperner's lemma



Two labelings for Sperner's lemma



Two labelings for Sperner's lemma



Theorem (Babson 2012)

Let T be a triangulation of the d-simplex $\triangle^d$ with two Sperner labelings λ_1, λ_2. For every choice of two nonnegative integers $d_1 + d_2 = d$, there exists $\sigma \in \mathsf{T}$ having for each i a rainbow d_i-face in λ_i.

A multilabeled version of Sperner's lemma

Theorem (Babson 2012)

Let T be a triangulation of the d-simplex $\triangle^d$ with m Sperner labelings $\lambda_1, \ldots, \lambda_m$. For every choice of m nonnegative integers $d_1 + \cdots + d_m = d$, there exists $\sigma \in \mathsf{T}$ having for each i a rainbow d_i-face in λ_i.

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Multilabeled Fan's lemma

Given a centrally symmetric triangulation T of $\mathcal{S}^d$, a Fan labeling $\lambda\colon V(\mathrm{T}) \to \mathbb{Z} \setminus \{0\}$ is a labeling s.t.

- $\lambda(u) + \lambda(v) \neq 0$ for all pairs of adjacent vertices u, v.
- $\lambda(-v) = -\lambda(v)$ for all $v \in V(\mathrm{T})$.

Theorem (Fan (1952))

Let T be a centrally symmetric triangulation of $\mathcal{S}^d$ with a Fan labeling λ. Then there is an alternating d-simplex.

Multilabeled Fan's lemma

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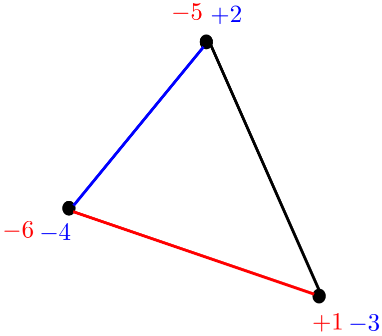
Theorem (M., Su (2019))

Let T be a centrally symmetric triangulation of $\mathcal{S}^d$ with m Fan labelings $\lambda_1, \ldots, \lambda_m$. For every choice of m nonnegative integers $d_1 + \cdots + d_m = d$, there exists $\sigma \in \mathsf{T}$ having for each i an alternating d_i-face in λ_i.

Multilabeled Fan's lemma



Theorem (M., Su (2019))

Let T *be a centrally symmetric triangulation of* $\mathcal{S}^d$ *with* m *Fan labelings* $\lambda_1, \ldots, \lambda_m$*. For every choice of* m *nonnegative integers* $d_1 + \cdots + d_m = d$*, there exists* $\sigma \in$ T *having for each* i *an alternating* d_i*-face in* λ_i*.*

Proof

Alishahi (2017) showed how to prove Fan's lemma from Tucker's lemma (or Borsuk-Ulam) in a direct way:

On each vertex σ of sd(T)*, put a label that records the size of the largest alternating face of σ.*

Here:

- Proof by contradiction.
- Define $\mu(\sigma) = \pm[d_1 + \cdots + d_{i^\star(\sigma)-1} + \mathrm{alt}_{\lambda_{i^\star(\sigma)}}(\sigma)]$, where
  - $\mathrm{alt}_{\lambda_i}(\sigma) =$ maximum number of vertices of an alternating face of σ.
  - $i^\star(\sigma) =$ smallest i such that $\mathrm{alt}_{\lambda_i(\sigma)} \leqslant d_i$.
- Apply Tucker lemma.

The most general form

$\mathrm{ind}(\mathsf{K}) := \min\left\{d\colon \mathsf{K} \to_{Z_2} \mathcal{S}^d\right\}$.

Theorem (M., Su (2019))

Let K be a free simplicial Z_2-complex with m Fan labelings $\lambda_1, \ldots, \lambda_m$. For every choice of m nonnegative integers $d_1 + \cdots + d_m = \mathrm{ind}(\mathsf{K})$, there exists $\sigma \in \mathsf{K}$ having for each i an alternating d_i-face in λ_i.

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Rainbow bipartite subgraphs

Graph $G = (V, E)$

$\mathrm{Hom}(K_2, G)$ is the poset s.t. (X, Y) is an element if

- $X, Y \neq \varnothing$
- $X \cap Y = \varnothing$
- $G[X, Y]$ is complete bipartite

with $(X, Y) \preccurlyeq (X', Y') \iff X \subseteq X'$ and $Y \subseteq Y'$.

Theorem (Simonyi, Tardif, Zsbán (2013))

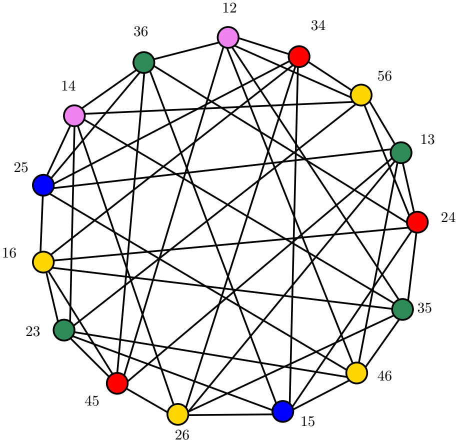
Any properly colored graph G contains a rainbow $K_{\lceil \frac{d}{2} \rceil + 1, \lfloor \frac{d}{2} \rfloor + 1}$ with $d = \mathrm{ind}(\mathrm{Hom}(K_2, G))$.

Strengthening of a theorem by Simonyi and Tardos (2007).

Example

Kneser graph $\mathrm{KG}(n,k) :=$
graph with $V = \binom{[n]}{k}$ and $E = \{AB \colon A \cap B = \varnothing\}$.

$\mathrm{KG}(6,2) =$



$d = \mathrm{ind}(\mathrm{Hom}(K_2, \mathrm{KG}(6,2))) = 2$, existence of rainbow $K_{\lceil \frac{d}{2} \rceil + 1, \lfloor \frac{d}{2} \rfloor + 1}$

Example

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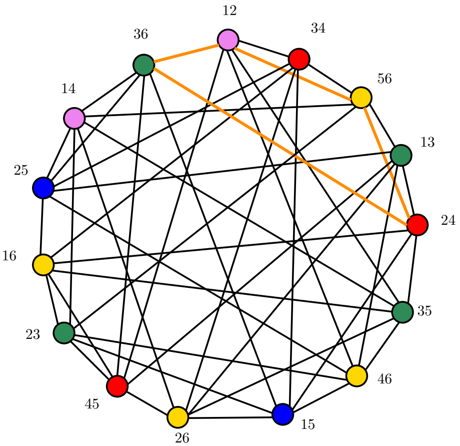


$d = \mathrm{ind}(\mathrm{Hom}(K_2, \mathrm{KG}(6,2))) = 2$, existence of rainbow $K_{\lceil \frac{d}{2} \rceil + 1, \lfloor \frac{d}{2} \rfloor + 1}$

Multirainbow bipartite subgraphs

Theorem

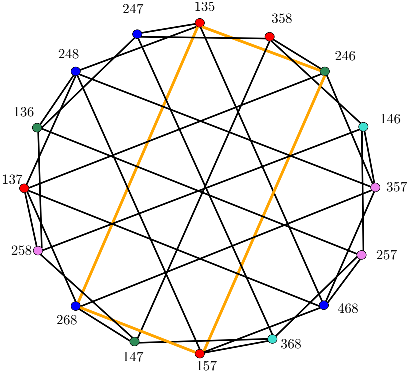
Consider a graph G with m proper colorings $c_1, \ldots, c_m$. For any choice of nonnegative integers s.t. $\sum_{i=1}^{m} d_i = \mathrm{ind}(\mathrm{Hom}(K_2, G))$, *there exists a complete bipartite subgraph that contains for each i a rainbow $K_{\lceil \frac{d_i}{2} \rceil + 1, \lfloor \frac{d_i}{2} \rfloor + 1}$ with respect to c_i.*

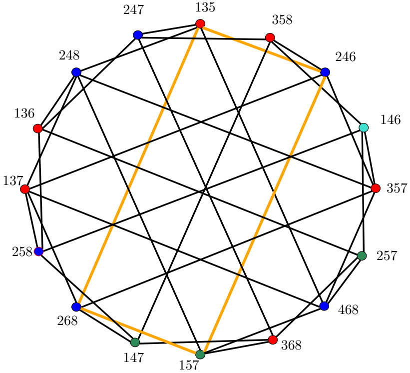
Example with two colorings

Schrijver graph $\mathrm{SG}(n,k) :=$

graph with $V = \{k\text{-stable sets of the } n\text{-cycle}\}$ and $E = \{AB\colon A \cap B = \varnothing\}$.

$\mathrm{SG}(8,3) =$





$d_1 + d_2 = 1 + 1 = \mathrm{ind}(\mathrm{Hom}(K_2, \mathrm{SG}(8,3))) = 2$

existence of rainbow $K_{\lceil \frac{d_i}{2} \rceil + 1, \lfloor \frac{d_i}{2} \rfloor + 1}$

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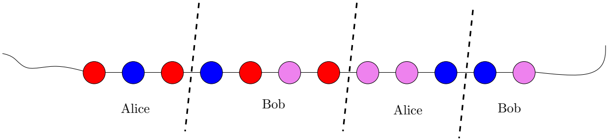
Fair splitting

k-splitting $=$ partition of $[0,1]$ into $k+1$ intervals $I_1,\dots,I_{k+1}$ (uses k cuts).

Given continuous measures $\mu_1,\dots,\mu_m$ on $[0,1]$, a k-splitting is fair if there is a partition A, B of $[k+1]$ s.t. $\mu_i\left(\bigcup_{\ell\in A} I_\ell\right) = \mu_i\left(\bigcup_{\ell\in B} I_\ell\right)$ for all i.

Theorem (Hobby, Rice (1965); Goldberg, West (1985))

For any choice of t continuous measures on $[0,1]$, there exists a fair t-splitting.



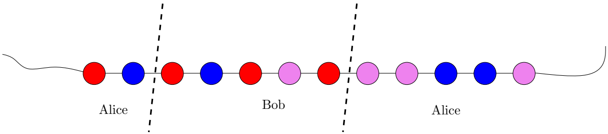
Balanced splitting

Given continuous measures $\mu_1, \ldots, \mu_m$ on $[0,1]$, a k-splitting is balanced if there is a partition A, B of $[k+1]$ and a $\gamma \in \mathbb{R}_+$ s.t.

- $\mu_i\left(\bigcup_{\ell \in A} I_\ell\right) = \mu_i\left(\bigcup_{\ell \in B} I_\ell\right) + \gamma$ for $\lceil m/2 \rceil$ indices i.
- $\mu_i\left(\bigcup_{\ell \in B} I_\ell\right) = \mu_i\left(\bigcup_{\ell \in A} I_\ell\right) + \gamma$ for the other $\lfloor m/2 \rfloor$ indices i.

Theorem (Pálvölgyi (2009) – special case)

For any choice of $t+1$ continuous measures on $[0,1]$, there exists a balanced t-splitting.



The theorem implies the fair splitting result ($\mu_m = 0$).

Balanced splitting of multiple necklaces

Theorem

Consider finite collections $\mathcal{M}_1, \ldots, \mathcal{M}_m$ of continuous measures on $[0,1]$. If $\sum_i |\mathcal{M}_i| = m + t$, then there exists a t-splitting that is balanced for all $\mathcal{M}_i$ simultaneously.

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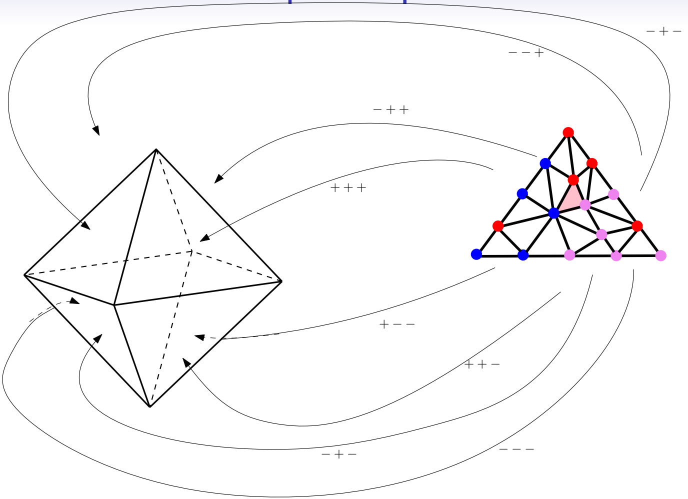
Fan's lemma implies Sperner's lemma



Fan with labels $\pm 1, \pm 2, \ldots, \pm(d+1)$
$\Rightarrow$ existence of an alternating simplex
$\Rightarrow$ existence of a rainbow simplex $\square$

Elementary proof of multilabeled Sperner's lemma

Theorem (Multilabeled Sperner lemma – reminder)

Let T *be a triangulation of the* d*-simplex* $\triangle^d$ *with* m *Sperner labelings* $\lambda_1, \ldots, \lambda_m$*. For every choice of* m *nonnegative integers* $d_1 + \cdots + d_m = d$*, there exists* $\sigma \in \mathsf{T}$ *having for each* i *a rainbow* d_i*-face in* λ_i*.*

Elementary proof (and constructive):

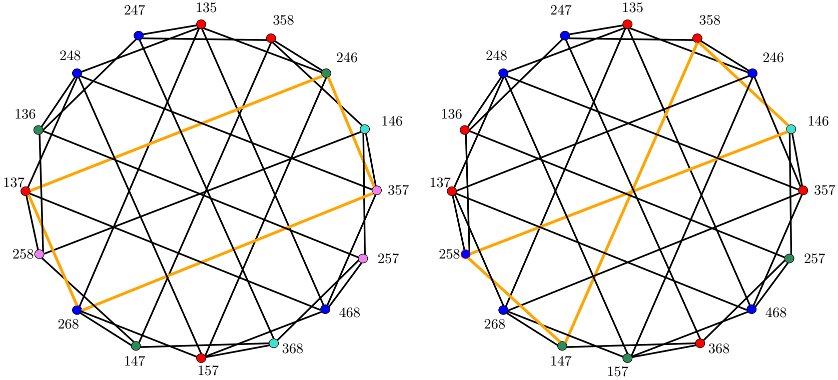
Multilabeled Fan with labels $\pm 1, \pm 2, \ldots, \pm(d+1)$

$\Rightarrow$ existence of a simplex with an alternating d_i-face for each λ_i

$\Rightarrow$ existence of a simplex with a rainbow d_i-face for each λ_i $\square$

Thank you.

Example – original theorem



$d = \mathrm{ind}(\mathrm{Hom}(K_2, \mathrm{SG}(8,3))) = 2$, existence of rainbow $K_{\lceil \frac{d}{2} \rceil + 1, \lfloor \frac{d}{2} \rfloor + 1}$