Colorful complete bipartite subgraphs in generalized Kneser graphs

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August 17th, 2018

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Any proper 3-coloring of the Petersen graph contains a $C_6$ colored cyclically with the 3 colors.
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Plan

- Chen’s theorem
- Generalization of Chen’s theorem
- Proof techniques and lemmas
- Applications and open questions
Chen’s theorem
The Petersen graph is also the graph with

\[ V = \binom{[5]}{2} \]

\[ E = \left\{ XY \in \left( \binom{V}{2} \right) : X \cap Y = \emptyset \right\} \]
Kneser graphs

The Petersen graph is the Kneser graph $KG(5, 2)$.

$KG(n, k)$ is the Kneser graph with

$$V = \binom{[n]}{k}$$
$$E = \left\{ XY \in \binom{V}{2} : X \cap Y = \emptyset \right\}$$

Theorem (Lovász 1978)

$\chi(KG(n, k)) = n - 2k + 2$. 
**Chen’s theorem**

**Theorem (Chen 2012)**

*Any proper coloring of $\text{KG}(n, k)$ with a minimum number of colors contains a $K_{n-2k+2}^*, n-2k+2$ with all colors on each side.*

\[ K_{t,t}^* = K_{t,t} \text{ minus a perfect matching.} \]

Petersen graph: $K_{3,3}^* = C_6$ and there always exists a...
Any proper $4$-coloring of $KG(6, 2)$ contains a $K^*_{4,4}$ with all $4$ colors on each side.
Any proper 4-coloring of KG(6, 2) contains a $K^*_4,4$ with all 4 colors on each side.
Generalization of Chen’s theorem
Generalized Kneser graphs

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph.

$KG(\mathcal{H})$ is the generalized Kneser graph with

$$
V = E(\mathcal{H})
$$

$$
E = \left\{ ef \in \binom{V}{2} : e \cap f = \emptyset \right\}
$$

$KG(n, k)$ obtained with $\mathcal{H} =$ complete $k$-uniform hypergraph on $n$ vertices.

Every simple graph is a generalized Kneser graph.
Dol’nikov’s theorem

Hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$. 2-colorability defect of $\mathcal{H}$:

$$\text{cd}_2(\mathcal{H}) = \left(\text{minimum number of vertices to remove so that the remaining hypergraph is 2-colorable}\right)$$

$$\text{cd}_2(\mathcal{H}) = \min |X| \text{ s.t. } (V(\mathcal{H}) \setminus X, \{ e \in E(\mathcal{H}) : e \cap X = \emptyset \}) \text{ is 2-colorable}$$

**Theorem (Dol’nikov 1993)**

$$\chi(KG(\mathcal{H})) \geq \text{cd}_2(\mathcal{H}).$$
Examples

When \( \mathcal{H} \) is the \( k \)-uniform complete hypergraph on \( n \) vertices: 
\[
\text{cd}_2(\mathcal{H}) = n - 2k + 2.
\]

When \( \mathcal{H} \) is a graph: \( \text{cd}_2(\mathcal{H}) = \) minimum of vertices to remove so that we get a bipartite graph.

\[
\text{has } \chi(\text{KG}(\mathcal{H})) = 4 \text{ and } \text{cd}_2(\mathcal{H}) = 2.
\]
Generalization of Chen’s theorem

Theorem (Alishahi-Hajiabolhassan-M. 2017)

Let $\mathcal{H}$ be a hypergraph with no singleton. If $\chi(KG(\mathcal{H})) = \text{cd}_2(\mathcal{H})$, then any proper coloring of $KG(\mathcal{H})$ with a minimum number of colors contains a $K_{\text{cd}_2(\mathcal{H}),\text{cd}_2(\mathcal{H})}^*$ with all colors on each side.

Example: $\chi(KG(\mathcal{H})) = \text{cd}_2(\mathcal{H}) = 4.$
Proof techniques and lemmas
Techniques

- Combinatorics
- Topological combinatorics
Case $\text{cd}_2(\mathcal{H}) = 1$

Let $\mathcal{H}$ be a hypergraph with no singleton. If $\chi(\text{KG}(\mathcal{H})) = \text{cd}_2(\mathcal{H}) = 1$, then any proper coloring of $\text{KG}(\mathcal{H})$ with a minimum number of colors contains a monochromatic $K_{1,1}^*$. 

If $\chi(\text{KG}(\mathcal{H})) = \text{cd}_2(\mathcal{H}) = 1$ means that any two edges of $\mathcal{H}$ intersect.

$\text{cd}_2(\mathcal{H}) = 1$ implies that there are at least two edges.
Case $\text{cd}_2(\mathcal{H}) = 1$

Let $\mathcal{H}$ be a hypergraph with no singleton. If $\chi(KG(\mathcal{H})) = \text{cd}_2(\mathcal{H}) = 1$, then $KG(\mathcal{H})$ has two non-adjacent vertices.
Case $cd_2(\mathcal{H}) = 1$

Let $\mathcal{H}$ be a hypergraph with no singleton. If $\chi(KG(\mathcal{H})) = cd_2(\mathcal{H}) = 1$, then $KG(\mathcal{H})$ has two non-adjacent vertices.

- $\chi(KG(\mathcal{H})) = 1$ means that any two edges of $\mathcal{H}$ intersect.
- $cd_2(\mathcal{H}) = 1$ implies that there are at least two edges. □
Case $cd_2(\mathcal{H}) = 2$

Let $\mathcal{H}$ be a hypergraph with no singleton. If $\chi(KG(\mathcal{H})) = cd_2(\mathcal{H}) = 2$, then any proper coloring of $KG(\mathcal{H})$ with a minimum number of colors contains a $K_{2,2}^*$ with two colors on each side.
Case $\text{cd}_2(\mathcal{H}) = 2$

Let $\mathcal{H}$ be a hypergraph with no singleton. If $\chi(KG(\mathcal{H})) = \text{cd}_2(\mathcal{H}) = 2$, then $KG(\mathcal{H})$ has two disjoint edges.
Case \( cd_2(\mathcal{H}) = 2 \)

Let \( \mathcal{H} \) be a hypergraph with no singleton. If \( \chi(KG(\mathcal{H})) = cd_2(\mathcal{H}) = 2 \), then \( KG(\mathcal{H}) \) has two disjoint edges.

Largest 2-colorable part of \( \mathcal{H} \)
colored with colors \( \{+, -\} \):

\[
\begin{array}{cccc}
+ & - & + & - & - \\
+ & + & + & - & - \\
+ & + & + & - & - \\
+ & + & + & - & - \\
\end{array}
\]

edge \( e_+ \in E(\mathcal{H}) \)

\[
\begin{array}{cccc}
+ & - & + & - & - \\
+ & - & - & + & + \\
+ & - & - & + & + \\
+ & - & - & + & + \\
\end{array}
\]

edge \( e_- \in E(\mathcal{H}) \)

\[
\begin{array}{cccc}
+ & - & + & - & - \\
+ & - & + & + & + \\
+ & - & + & + & + \\
+ & - & + & + & + \\
\end{array}
\]

edge \( f_+ \in E(\mathcal{H}) \)

\[
\begin{array}{cccc}
+ & - & + & - & - \\
+ & - & + & - & - \\
+ & - & + & - & - \\
+ & - & + & - & - \\
\end{array}
\]

edge \( f_- \in E(\mathcal{H}) \)
The topological method

The topological method in a nutshell

∃ proper coloring \( c \) of \( G = (V, E) \) with \( t \) colors

\[ \exists Z_2\text{-complex } L(G) \text{ and } Z_2\text{-equivariant map } \phi : L(G) \rightarrow S^{f(t)}. \]

Obstruction (e.g., the Borsuk-Ulam theorem) \( \Rightarrow \) lower bound on \( t \).

Proof (Ziegler 2001, Matoušek 2003) of Dol’nikov’s theorem

\[ \chi(KG(H)) \geq \text{cd}_2(H): \]

\[ \exists \text{ simplicial } Z_2\text{-map } \phi : \text{sd } Z_2^n \rightarrow Z_2^{*(n-\text{cd}_2(H)+t)} \]

where \( n = |V(H)| \), conclude with Tucker’s lemma:

\[ n \leq n - \text{cd}_2(H) + t. \]
\[\phi(\mathbf{x}) = \begin{cases} 
\pm (n - cd_2(\mathcal{H}) + \max c(S)) & \text{for } S \in E(\mathcal{H}) \text{ and } (S \subseteq \mathbf{x}^+ \text{ or } S \subseteq \mathbf{x}^-) \\
\pm (|\mathbf{x}^+| + |\mathbf{x}^-|) & \text{if such } S \text{ does not exist.}
\end{cases}\]

\[\mathbf{x}^+ = \{i \in [n]: x_i = +\} \quad \text{and} \quad \mathbf{x}^- = \{i \in [n]: x_i = -\}\]
Fan’s lemma

Replace Tucker’s lemma by

**Lemma (Fan’s lemma)**

Let $T$ be a centrally symmetric triangulation of a $d$-sphere. For every simplicial $\mathbb{Z}_2$-map $\phi: T \to \mathbb{Z}_2^{\infty}$, there exists an alternating $d$-simplex.

An alternating simplex has an ordering of its vertices $v_0, \ldots, v_d$ s.t.

$0 < +\phi(v_0) < -\phi(v_1) < +\phi(v_2) < \cdots < (-1)^d \phi(v_d)$.

**Theorem (Fan 1982, Simonyi-Tardos 2006)**

There exists a colorful bipartite complete subgraph $K_{\lceil cd_2(\mathcal{H})/2 \rceil, \lfloor cd_2(\mathcal{H})/2 \rfloor}$ in any proper coloring of $KG(\mathcal{H})$.

Strengthening for graphs with $\chi(KG(\mathcal{H})) = cd_2(\mathcal{H})$ (Spencer-Su 2005, Simonyi-Tardos 2007).
Chen’s lemma

Replace Fan’s lemma by

Lemma (Chen 2012)

Consider an order-preserving $\mathbb{Z}_2$-map

$$\phi : \{+, -, 0\}^n \setminus \{0\} \rightarrow \{\pm 1, \ldots, \pm n\}.$$ 

Suppose moreover that there is a $\gamma \in [n]$ such that when $x \prec y$, at most one of $|\phi(x)|$ and $|\phi(y)|$ is equal to $\gamma$. Then there are two chains

$$x_1 \preceq \cdots \preceq x_n \quad \text{and} \quad y_1 \preceq \cdots \preceq y_n$$

such that

$$\phi(x_i) = (-1)^i i \quad \text{for all } i \quad \text{and} \quad \phi(y_i) = (-1)^i i \quad \text{for } i \neq \gamma$$

and such that $x_\gamma = -y_\gamma$.

Proved with the help of Fan’s lemma.
Applications and open questions
Circular chromatic number

Graph $G = (V, E)$

$(p, q)$-coloring: $c : V \to [p]$ such that $q \leq |c(u) - c(v)| \leq p - q$ when $uv \in E$.

Circular chromatic number: $\chi_c(G) = \inf\{p/q : \exists (p, q)\text{-coloring}\}$. 
Circular chromatic number

Graph $G = (V, E)$

$(p, q)$-coloring: $c : V \rightarrow [p]$ such that $q \leq |c(u) - c(v)| \leq p - q$ when $uv \in E$.

Circular chromatic number: $\chi_c(G) = \inf\{p/q : \exists (p, q)$-coloring$\}$.

Properties.

- The infimum is in fact a minimum.
- $\chi(G) = \lceil \chi_c(G) \rceil$.
- Computing $\chi_c(G)$: NP-hard.
When does $\chi_c(G) = \chi(G)$ hold?

Question that has received considerable attention (Zhu 2001).

**Theorem (Simonyi-Tardos 2006)**

$\chi(G) = \chi_c(G)$ when $G$ is “topologically $\chi(G)$-chromatic” and $\chi(G)$ is even.

**Lemma (Folklore)**

If every proper $t$-coloring of a $t$-chromatic graph $G$ contains a $K_{t,t}^*$ with all colors on each side, then $\chi(G) = \chi_c(G)$.

**Theorem (Alishahi-Hajiabolhassan-M. 2017)**

If $\chi(KG(H)) = cd_2(H)$, then $\chi(G) = \chi_c(G)$.

Categorical product

Theorem (Alishahi-Hajiabolhassan-M. 2017)

Let $\mathcal{H}_1, \ldots, \mathcal{H}_s$ be hypergraphs with no singleton and such that $\chi(KG(\mathcal{H}_i)) = \text{cd}_2(\mathcal{H}_i)$ for all $i$. Let $t = \min_i \text{cd}_2(\mathcal{H}_i)$.

Then any proper coloring of $KG(\mathcal{H}_1) \times \cdots \times KG(\mathcal{H}_s)$ with $t$ colors contains a $K^*_{t,t}$ with all colors on each side.

Consequence: for such hypergraphs

$$
\chi(KG(\mathcal{H}_1) \times \cdots \times KG(\mathcal{H}_s)) = \chi_c(KG(\mathcal{H}_1) \times \cdots \times KG(\mathcal{H}_s))$

$$
= \min_i \chi(KG(\mathcal{H}_i))) = \min_i \chi_c(KG(\mathcal{H}_i))) = \min_i (\text{cd}(\mathcal{H}_i)).$$

They satisfy Hedetniemi’s conjecture and Hedetniemi’s conjecture for the circular coloring (Zhu 1992).
Examples of hypergraphs $\mathcal{H}$ with $\chi(KG(\mathcal{H})) = cd_2(\mathcal{H})$

Examples of hypergraphs $\mathcal{H}$ for which

$$\chi(KG(\mathcal{H})) = cd_2(\mathcal{H}).$$

(⋆)

1. Let $G$ be a triangle-free graph and choose $k \geq \alpha(G)$. Denote by $G(k)$ the join of $G$ with the disjoint union of $k$ triangles.
Then $\mathcal{H} = G(k)$ satisfies (⋆).

2. Let $A$ and $B$ be two disjoint sets, with $|A| \geq 2k - 1$ and $|B| \geq 1$.
Then $\mathcal{H} = \binom{A}{k} \cup \left\{ \{i, j\} : i \in A, j \in B \right\} \cup \binom{B}{k}$ satisfies (⋆).

(Example due to Simonyi)
These two problems are NP-hard:

- Deciding $\chi(G) = \chi_c(G)$ (Guichard, 1993).
- Deciding $\chi(KG(\mathcal{H})) = cd_2(\mathcal{H})$ (M.-Mizrahi, 2018).

**Theorem (Hatami-Tusserkani, 2004)**

*Deciding $\chi(G) = \chi_c(G)$ remains NP-hard when $\chi(G)$ is known.*

What is the complexity of deciding $\chi(KG(\mathcal{H})) = cd_2(\mathcal{H})$ when $\chi(KG(\mathcal{H}))$ is known?
Thank you