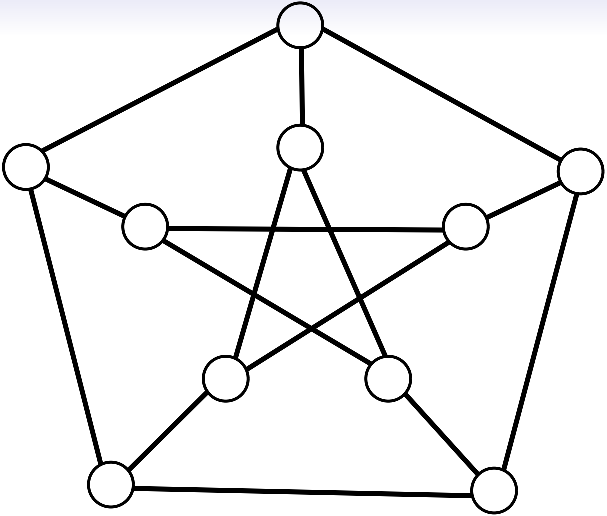
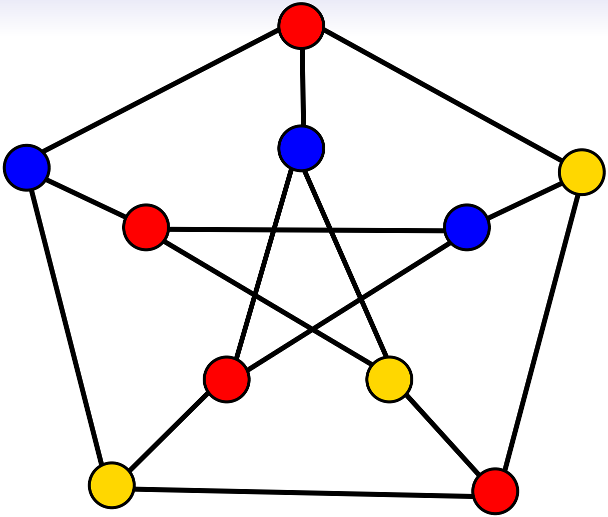
Colorful complete bipartite subgraphs in generalized Kneser graphs

Frédéric Meunier

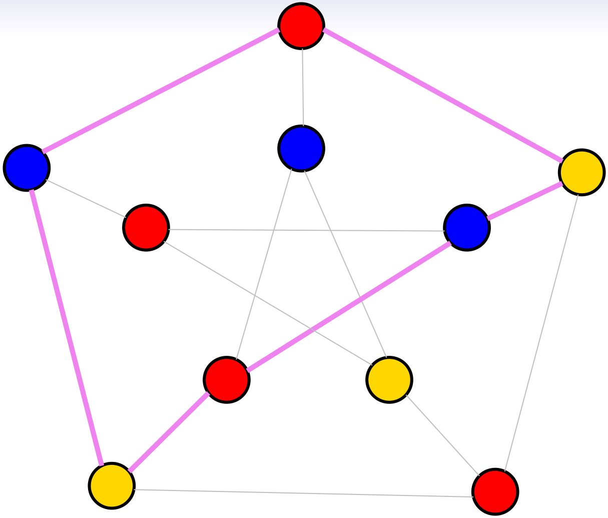
August 17th, 2018

Joint work with Meysam Alishahi and Hossein Hajiabolhassan

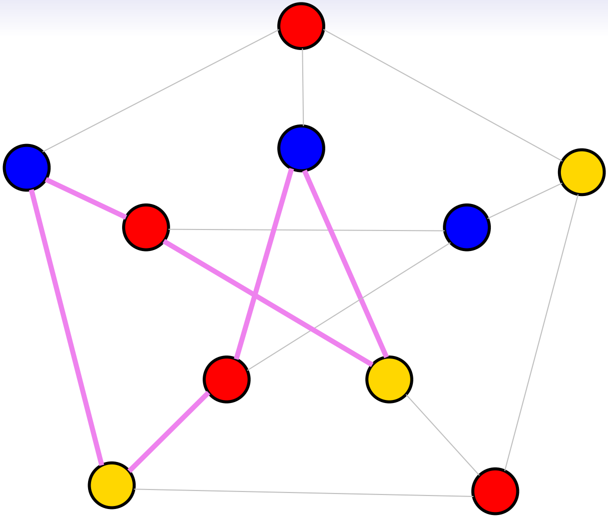




Any proper 3-coloring of the Petersen graph contains a C_6 colored cyclically with the 3 colors.



Any proper 3-coloring of the Petersen graph contains a C_6 colored cyclically with the 3 colors.



Any proper 3-coloring of the Petersen graph contains a C_6 colored cyclically with the 3 colors.

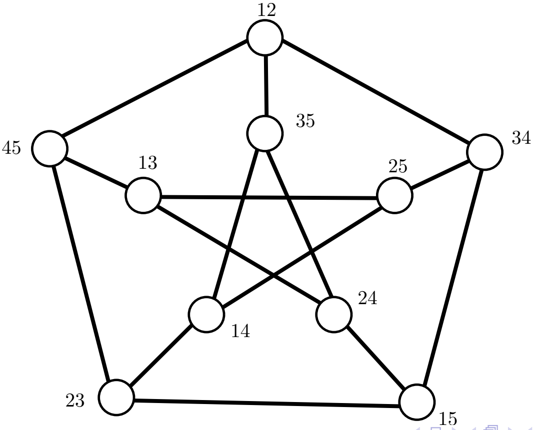
Plan

- Chen's theorem
- Generalization of Chen's theorem
- Proof techniques and lemmas
- Applications and open questions

Chen’s theorem

The Petersen graph is also the graph with

$$\begin{aligned} V &= \binom{[5]}{2} \\ E &= \left\{ XY \in \binom{V}{2} : X \cap Y = \varnothing \right\} \end{aligned}$$



Kneser graphs

The Petersen graph is the Kneser graph $\mathrm{KG}(5, 2)$.

$\mathrm{KG}(n, k)$ is the Kneser graph with

$$\begin{aligned} V &= \binom{[n]}{k} \\ E &= \left\{ XY \in \binom{V}{2} : X \cap Y = \varnothing \right\} \end{aligned}$$

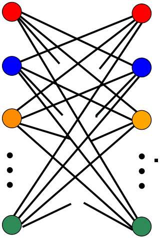
Theorem (Lovász 1978)

$\chi(\mathrm{KG}(n, k)) = n - 2k + 2.$

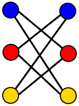
Chen's theorem

Theorem (Chen 2012)

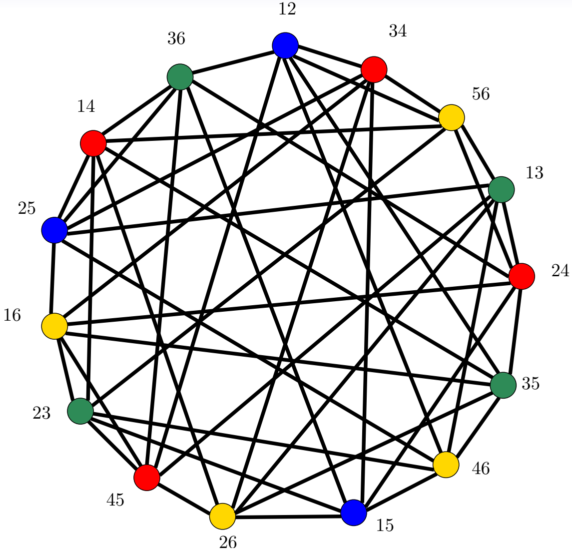
Any proper coloring of $\mathrm{KG}(n,k)$ *with a minimum number of colors contains a* $K^*_{n-2k+2,n-2k+2}$ *with all colors on each side.*



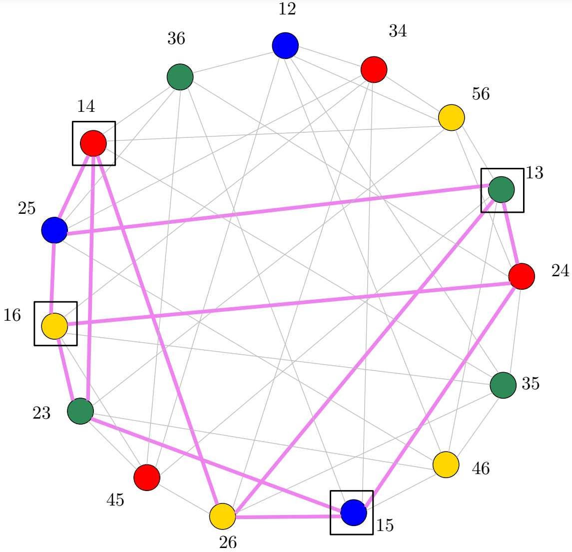
$K^*_{t,t} = K_{t,t}$ minus a perfect matching.

Petersen graph: $K^*_{3,3} = C_6$ and there always exists a .

KG(6, 2):



KG(6, 2):



Any proper 4-coloring of KG(6, 2) *contains a* $K^*_{4,4}$ *with all 4 colors on each side.*

Generalization of Chen’s theorem

Generalized Kneser graphs

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph.

KG($\mathcal{H}$) is the generalized Kneser graph with

$$\begin{aligned} V &= E(\mathcal{H}) \\ E &= \left\{ ef \in \binom{V}{2} : e \cap f = \varnothing \right\} \end{aligned}$$

KG(n, k) obtained with $\mathcal{H}$ = complete k-uniform hypergraph on n vertices.

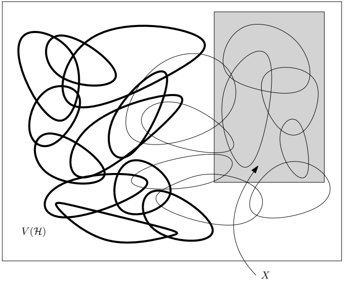
Every simple graph is a generalized Kneser graph.

Dol'nikov's theorem

Hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$. 2-colorability defect of $\mathcal{H}$:

$\mathrm{cd}_2(\mathcal{H}) = \begin{pmatrix}\text{minimum number of vertices to remove so that the re-}\\ \text{maining hypergraph is 2-colorable}\end{pmatrix}$

$\mathrm{cd}_2(\mathcal{H}) = \min |X|$ s.t. $(V(\mathcal{H}) \setminus X, \{e \in E(\mathcal{H}) \colon e \cap X = \varnothing\})$ is 2-colorable



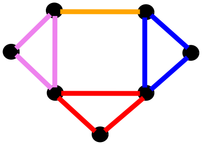
Theorem (Dol'nikov 1993)

$\chi(\mathrm{KG}(\mathcal{H})) \geqslant \mathrm{cd}_2(\mathcal{H})$.

Examples

When $\mathcal{H}$ is the k-uniform complete hypergraph on n vertices: $\mathrm{cd}_2(\mathcal{H}) = n - 2k + 2$.

When $\mathcal{H}$ is a graph: $\mathrm{cd}_2(\mathcal{H}) =$ minimum of vertices to remove so that we get a bipartite graph.



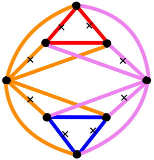
has $\chi(\mathrm{KG}(\mathcal{H})) = 4$ and $\mathrm{cd}_2(\mathcal{H}) = 2$.

Generalization of Chen's theorem

Theorem (Alishahi-Hajiabolhassan-M. 2017)

Let $\mathcal{H}$ be a hypergraph with no singleton.
If $\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H})$, then any proper coloring of $\mathrm{KG}(\mathcal{H})$ with a minimum number of colors contains a $K^*_{\mathrm{cd}_2(\mathcal{H}),\mathrm{cd}_2(\mathcal{H})}$ with all colors on each side.

Example:



$\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H}) = 4.$

Proof techniques and lemmas

Techniques

- Combinatorics
- Topological combinatorics

Case $\mathrm{cd}_2(\mathcal{H}) = 1$

Let $\mathcal{H}$ be a hypergraph with no singleton.
If $\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H}) = 1$, then any proper coloring of $\mathrm{KG}(\mathcal{H})$ with a minimum number of colors contains a monochromatic $K^*_{1,1}$.

Case $\mathrm{cd}_2(\mathcal{H}) = 1$

Let $\mathcal{H}$ be a hypergraph with no singleton.
If $\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H}) = 1$, *then* $\mathrm{KG}(\mathcal{H})$ *has two non-adjacent vertices.*

Case $\mathrm{cd}_2(\mathcal{H}) = 1$

Let $\mathcal{H}$ be a hypergraph with no singleton.
If $\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H}) = 1$, then $\mathrm{KG}(\mathcal{H})$ has two non-adjacent vertices.

- $\chi(\mathrm{KG}(\mathcal{H})) = 1$ means that any two edges of $\mathcal{H}$ intersect.
- $\mathrm{cd}_2(\mathcal{H}) = 1$ implies that there are at least two edges. □

Case $\mathrm{cd}_2(\mathcal{H}) = 2$

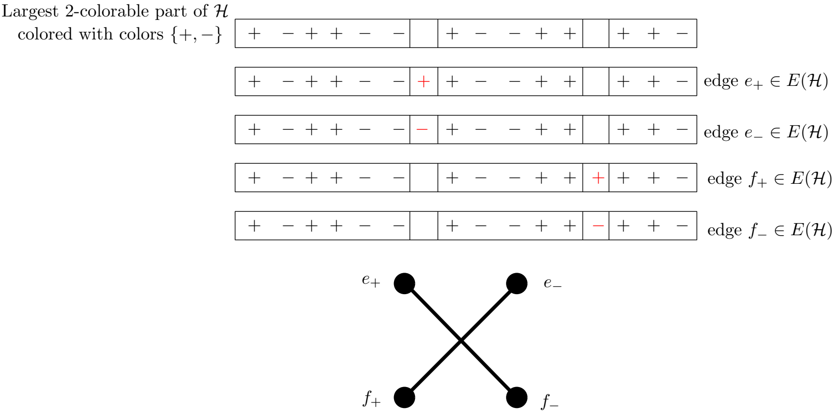
Let $\mathcal{H}$ be a hypergraph with no singleton.
If $\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H}) = 2$, then any proper coloring of $\mathrm{KG}(\mathcal{H})$ with a minimum number of colors contains a $K^*_{2,2}$ with two colors on each side.

Case $\mathrm{cd}_2(\mathcal{H}) = 2$

Let $\mathcal{H}$ be a hypergraph with no singleton.
If $\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H}) = 2$, then $\mathrm{KG}(\mathcal{H})$ has two disjoint edges.

Case $\mathrm{cd}_2(\mathcal{H}) = 2$

Let $\mathcal{H}$ be a hypergraph with no singleton.
If $\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H}) = 2$, then $\mathrm{KG}(\mathcal{H})$ has two disjoint edges.



The topological method

The topological method in a nutshell

$\exists$ proper coloring c of $G = (V, E)$ with t colors

$\Longrightarrow$

$\exists$ Z_2-complex $\mathrm{L}(G)$ and Z_2-equivariant map $\phi\colon \mathrm{L}(G) \longrightarrow \mathcal{S}^{f(t)}$.

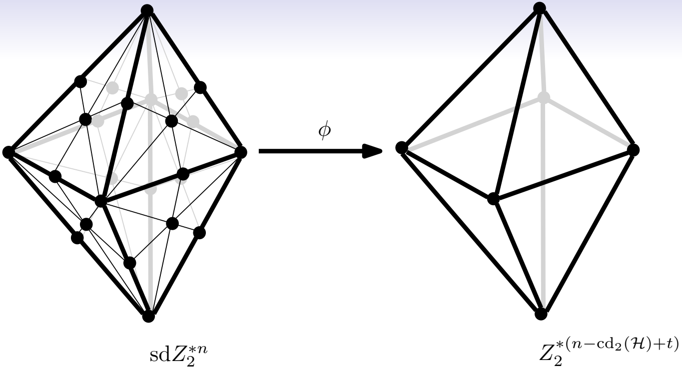
Obstruction (e.g., the Borsuk-Ulam theorem) $\Rightarrow$ lower bound on t.

Proof (Ziegler 2001, Matoušek 2003) of Dol'nikov's theorem $\chi(\mathrm{KG}(\mathcal{H})) \geqslant \mathrm{cd}_2(\mathcal{H})$:

$$\exists \text{ simplicial } Z_2\text{-map } \phi : \mathrm{sd}\, Z_2^{*n} \longrightarrow Z_2^{*(n-\mathrm{cd}_2(\mathcal{H})+t)}$$

where $n = |V(\mathcal{H})|$, conclude with Tucker's lemma:

$$n \leqslant n - \mathrm{cd}_2(\mathcal{H}) + t.$$



$$\boldsymbol{x} \in \{+,-,0\}^n \setminus \{\mathbf{0}\} \longmapsto \phi(\boldsymbol{x}) \in \big\{ \pm 1, \pm 2, \ldots, \pm(n - \mathrm{cd}_2(\mathcal{H}) + t)\big\}$$

$$\boldsymbol{x}^+ = \{i \in [n] \colon x_i = +\} \qquad \text{and} \qquad \boldsymbol{x}^- = \{i \in [n] \colon x_i = -\}$$

$$\phi(\boldsymbol{x}) = \begin{cases} \pm\,(n - \mathrm{cd}_2(\mathcal{H}) + \max c(S)) & \text{for } S \in E(\mathcal{H}) \text{ and } (S \subseteq \boldsymbol{x}^+ \text{ or } S \subseteq \boldsymbol{x}^-) \\ \pm\left(|\boldsymbol{x}^+| + |\boldsymbol{x}^-|\right) & \text{if such } S \text{ does not exist.} \end{cases}$$

Fan's lemma

Replace Tucker's lemma by

Lemma (Fan's lemma)

Let T *be a centrally symmetric triangulation of a d-sphere. For every simplicial Z_2-map $\phi\colon \mathrm{T} \to Z_2^{*\infty}$, there exists an alternating d-simplex.*

An alternating simplex has an ordering of its vertices $v_0, \ldots, v_d$ s.t.
$0 < +\phi(v_0) < -\phi(v_1) < +\phi(v_2) < \cdots < (-1)^d\phi(v_d)$.

Theorem (Fan 1982, Simonyi-Tardos 2006)

There exists a colorful bipartite complete subgraph $K_{\lceil \mathrm{cd}_2(\mathcal{H})/2\rceil, \lfloor \mathrm{cd}_2(\mathcal{H})/2\rfloor}$ in any proper coloring of $\mathrm{KG}(\mathcal{H})$.

Strengthening for graphs with $\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H})$ (Spencer-Su 2005, Simonyi-Tardos 2007).

Chen's lemma

Replace Fan's lemma by

Lemma (Chen 2012)

Consider an order-preserving $\mathbb{Z}_2$*-map* $\phi : \{+,-,0\}^n \setminus \{\mathbf{0}\} \to \{\pm 1, \ldots, \pm n\}$. *Suppose moreover that there is a* $\gamma \in [n]$ *such that when* $\boldsymbol{x} \prec \boldsymbol{y}$, *at most one of* $|\phi(\boldsymbol{x})|$ *and* $|\phi(\boldsymbol{y})|$ *is equal to* γ. *Then there are two chains*

$$\boldsymbol{x}_1 \preccurlyeq \cdots \preccurlyeq \boldsymbol{x}_n \quad \text{and} \quad \boldsymbol{y}_1 \preccurlyeq \cdots \preccurlyeq \boldsymbol{y}_n$$

such that

$$\phi(\boldsymbol{x}_i) = (-1)^i i \quad \text{for all } i \qquad \text{and} \qquad \phi(\boldsymbol{y}_i) = (-1)^i i \quad \text{for } i \neq \gamma$$

and such that $\boldsymbol{x}_\gamma = -\boldsymbol{y}_\gamma$.

Proved with the help of Fan's lemma.

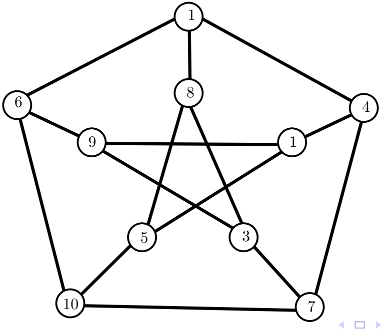
Applications and open questions

Circular chromatic number

Graph $G = (V, E)$

(p, q)-coloring: $c : V \to [p]$ such that $q \leqslant |c(u) - c(v)| \leqslant p - q$ when $uv \in E$.

Circular chromatic number: $\chi_c(G) = \inf\{p/q : \exists (p, q)\text{-coloring}\}$.



Circular chromatic number

Graph $G = (V, E)$

(p, q)-coloring: $c : V \to [p]$ such that $q \leqslant |c(u) - c(v)| \leqslant p - q$ when $uv \in E$.

Circular chromatic number: $\chi_c(G) = \inf\{p/q \colon \exists (p, q)\text{-coloring}\}$.

Properties.

- The infimum is in fact a minimum.
- $\chi(G) = \lceil \chi_c(G) \rceil$.
- Computing $\chi_c(G)$: NP-hard.

When does $\chi_c(G) = \chi(G)$ hold?

Question that has received a considerable attention (Zhu 2001).

Theorem (Simonyi-Tardos 2006)

$\chi(G) = \chi_c(G)$ when G is "topologically $\chi(G)$-chromatic" and $\chi(G)$ is even.

Lemma (Folklore)

If every proper t-coloring of a t-chromatic graph G contains a $K^*_{t,t}$ with all colors on each side, then $\chi(G) = \chi_c(G)$.

Theorem (Alishahi-Hajiabolhassan-M. 2017)

If $\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H})$, then $\chi(G) = \chi_c(G)$.

Case of $\mathrm{KG}(n,k)$: Chen (2012). Partial results by Hajiabolhassan-Zhu (2003), M. (2005).

Categorical product

Theorem (Alishahi-Hajiabolhassan-M. 2017)

Let $\mathcal{H}_1, \ldots, \mathcal{H}_s$ be hypergraphs with no singleton and such that $\chi(\mathrm{KG}(\mathcal{H}_i)) = \mathrm{cd}_2(\mathcal{H}_i)$ for all i. Let $t = \min_i \mathrm{cd}_2(\mathcal{H}_i)$.
Then any proper coloring of $\mathrm{KG}(\mathcal{H}_1) \times \cdots \times \mathrm{KG}(\mathcal{H}_s)$ with t colors contains a $K^*_{t,t}$ with all colors on each side.

Consequence: for such hypergraphs

$$\chi(\mathrm{KG}(\mathcal{H}_1) \times \cdots \times \mathrm{KG}(\mathcal{H}_s)) = \chi_c(\mathrm{KG}(\mathcal{H}_1) \times \cdots \times \mathrm{KG}(\mathcal{H}_s))$$

$$= \min_i(\chi(\mathrm{KG}(\mathcal{H}_i))) = \min_i(\chi_c(\mathrm{KG}(\mathcal{H}_i))) = \min_i(\mathrm{cd}(\mathcal{H}_i)).$$

They satisfy Hedetniemi's conjecture and Hedetniemi's conjecture for the circular coloring (Zhu 1992).

Examples of hypergraphs $\mathcal{H}$ with $\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H})$

Examples of hypergraphs $\mathcal{H}$ for which

$$\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H}). \qquad (\star)$$

1. Let G be a triangle-free graph and choose $k \geqslant \alpha(G)$. Denote by $G(k)$ the join of G with the disjoint union of k triangles.
   Then $\mathcal{H} = G(k)$ satisfies $(\star)$.

2. Let A and B be two disjoint sets, with $|A| \geqslant 2k - 1$ and $|B| \geqslant 1$.
   Then $\mathcal{H} = \binom{A}{k} \cup \big\{\{i,j\} : i \in A,\ j \in B\big\} \cup \binom{B}{k}$ satisfies $(\star)$.

(Example due to Simonyi)

Complexity aspects

These two problems are NP-hard:

- Deciding $\chi(G) = \chi_c(G)$ (Guichard, 1993).
- Deciding $\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H})$ (M.-Mizrahi, 2018).

Theorem (Hatami-Tusserkani, 2004)

Deciding $\chi(G) = \chi_c(G)$ remains NP-hard when $\chi(G)$ is known.

What is the complexity of deciding $\chi(\mathrm{KG}(\mathcal{H})) = \mathrm{cd}_2(\mathcal{H})$ when $\chi(\mathrm{KG}(\mathcal{H}))$ is known?

Thank you