Existence results in topological combinatorics with open complexity status

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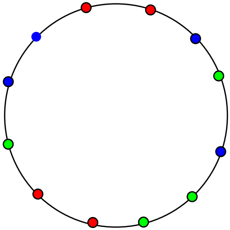
Recent Advances on Total Search Problems

Plan

Statement

Theorem Haxell 1995

In a colored graph with maximum degree Δ, there always exists an independent set intersecting every color class of size at least 2Δ.



2Δ is tight.

Proofs

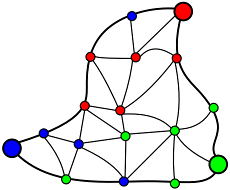
Original proof: based on an alternating path argument
$\Longrightarrow$ PLS ?

Short proof (Aharoni, Berger, and Ziv 2007): based on a Sperner-like topological lemma due to Meshulam.
$\Longrightarrow$ PPAD ?

Meshulam's lemma

Lemma Meshulam 2001

Consider a simplicial complex K with colored vertices. If for any subset C of the colors the corresponding induced subcomplex $\mathrm{K}[C]$ is $(|C|-2)$-connected, then there exists a simplex with all colors.



Non-constructive proof (connectivity...)

Many other "TFNP" applications: colorful Carathéodory, large rainbow matchings, etc.

Known results

Theorem Graf, Haxell 2020

Consider a colored graph G with (fixed) maximum degree Δ. An independent set intersecting every color class of size at least $2\Delta + 1$ can be found in time polynomial in $|V(G)|$.

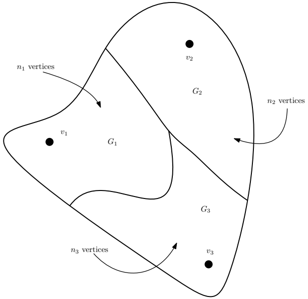
Complexity status completely open for the original version with 2Δ.

Plan

Statement

Theorem Győri 1976, Lovász 1977

Let $k \geqslant 2$ be an integer, let G be a k-connected graph on n vertices, let $v_1, v_2, \ldots, v_k$ be distinct vertices of G and let $n_1, n_2, \ldots, n_k$ be positive integers with $n_1 + n_2 + \cdots + n_k = n$. Then G has disjoint connected subgraphs $G_1, G_2, \ldots, G_k$ such that for $i = 1, 2, \ldots, k$ the graph G_i has n_i vertices and $v_i \in V(G_i)$.



Proofs

Constructive proof (Győri 1976)

Topological proof (Lovász 1977): based on a homological result
$\Longrightarrow$ PPAD ?

The homological result by Lovász

Theorem Lovász 1977

Let D be a digraph with a vertex r that cannot be separated from any other vertex by removing less than $k \geqslant 2$ vertices. Then we have $\widetilde{H}_i(\mathsf{A}, \mathbb{Z}) = 0$ for $i = 0, \ldots, k-2$, where A is the "arborescence complex" relative to r.

Known results and questions

Theorem Chandran, Cheung, Issac 2018

Finding the partition of the Győri–Lovász theorem is in PLS.

Proof by an adaptation of the original proof by Győri

Open question Chandran, Cheung, Issac 2018

Is the problem PLS-complete?

The problem is known to be polynomial for $k = 2, 3$.

Is the problem in PPAD?

Plan

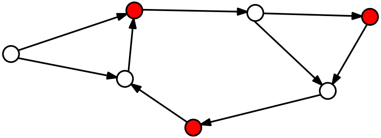
Statement

A simple directed graph is clique-acyclic if every clique is acyclic.

kernel = subset of vertices that is independent and absorbing

Theorem Boros, Gurvich 1996

Every clique-acyclique orientation of a perfect graph has a kernel.



Proofs

All known proofs are topological:

- **Original proof**: relies on advanced results in game theory (but ultimately on Scarf's lemma?)
- **Short proof** (Aharoni, Holzman 1998): strong fractional kernels + Scarf's lemma
- **Short and simpler proof** (Király, Pap 2009): Sperner's lemma applied on the polar of $P = \mathrm{STAB}(G) - \mathbb{R}^n_+$

$\Longrightarrow$ PPAD ?

Issues: Scarf's lemma and P use matrices with rows indexed by all (maximal) cliques

Strong fractional kernels

Let $D = (V, A)$ be a digraph.

strong fractional kernel $= f\colon V \to \mathbb{R}_+$ s.t.

- for each clique K, we have $\sum_{u \in K} f(u) \leqslant 1$ (fractional independence)
- for each vertex v, there is a clique $K \subseteq N^+[v]$ with $\sum_{u \in K} f(u) \geqslant 1$ (strong fractional absorbance)

Theorem Aharoni, Holzman 1998

Every clique acyclic orientation of a simple digraph has a strong fractional kernel.

Known result

Theorem Kintali et al. 2013

Finding a strong fractional kernel is PPAD-hard.

Proof by reduction from Preference game, itself being reduced from 3-D Brouwer

The problem of finding a kernel in a perfect graph is in TFNP. Is it in PPAD?

Theorem Ayumi, M., Pass-Lanneau 2020

Finding a kernel in a claw-free perfect graph can be done in polynomial time.

Plan

Statement

Let $G = (V, E)$ be a graph.

A fractional matching for $F \subseteq E$ is a map $f : F \to \mathbb{R}_+$ s.t. $\sum_{e \in \delta(v) \cap F} f(e) \leqslant 1$.

size of a fractional matching $= \sum_{e \in F} f(e)$

$\nu^*(F) =$ maximal size of a fractional matching for F

rainbow fractional matching = fractional matching whose support intersects each E_i at most once

Theorem Aharoni, Holzman, Jiang 2019

Let n be an integer or half-integer. Every partition $E_1, \dots, E_{2n}$ of the edges such that $\nu^*(E_i) \geqslant n$ for all i has a rainbow fractional matching of size n.

Proof

The proof uses the following topological result.

d-Leray = trivial homology over $\mathbb{Q}$ in dimension d and above

Theorem Kalai, Meshulam 2005

Let K be a d-Leray simplicial complex and let M be a matroid with rank function ρ, both on the same vertex set V. If M is a subcomplex of K, then there exists $\tau \in \mathsf{K}$ such that $\rho(V \setminus \tau) \leqslant d$.

$\Longrightarrow$ PPAD ?

Many other "TFNP" applications: colorful results in discrete geometry, rainbow matchings, intersection matroids and oriented matroids, etc.

Plan

Statement: good

Players dividing a cake, with closed preferences: the “limit” of a sequence preferred parts is also preferred.

Cake identified with $[0, 1]$; parts identified with subintervals.

Theorem Stromquist 1980, Woodall 1980

No matter how many players there are, if the cake is good, then there is always an envy-free division.

Statement: bad and mixed

Theorem Su 1999

No matter how many players there are, if the cake is bad, then there is always an envy-free division.

Theorem Avvakumov, Karasev 2021

Suppose that the number of players is a prime power. Then there is always an envy-free division. (No assumption on the cake.)

case with three players proved by Segal-Halevi (2018), with a prime number of players proved by M., Zerbib (2019)

Proofs

Original proof for good cakes: Brouwer fixed point

Elementary proof for good/bad cakes (Su 1999): Sperner's lemma

$\Longrightarrow$ PPAD

Original proofs for "mixed" cakes: degree argument (involved)

Shorter proof for "mixed" cakes (Jojić, Panina, Živaljević 2019): Volovikov's theorem

Known results for good/bad cakes

Theorem Deng, Qi, Saberi 2012

Envy-free division of a good cake is PPAD-complete when the (fixed) number of players is at least three.

(Model: preferences given by polynomial time functions)

Under Lipschitz assumption, FPTAS for three players

Volovikov's theorem

Theorem Volovikov 1996

Let G be the additive group $\left((\mathbb{Z}_p)^k, +\right)$ and let X and Y be two topological spaces on which G acts in a fixed-point free way. If X is d-connected and Y is a d-dimensional sphere, then there is no G-equivariant map $X \to Y$.

$\Longrightarrow$ PPA-p^k ?

Issue (?):

- PPA-p^k = PPA-p (Hollender (2021))
- there is a version of Dold's theorem that is PPA-p-complete
- Dold's theorem enough to prove envy-free cake division in the mixed case?

Equipartition of a segment

$\mathcal{I} = \{[a, b) : 0 \leqslant a \leqslant b \leqslant 1\}$.

Theorem Avvakumov and Karasev 2022

Let $f : \mathcal{I} \to \mathbb{R}$ be a continuous map such that $f(\varnothing) = 0$ for all $a \in [0, 1]$. Then for every positive integer n, there exist $0 \leqslant x_1 \leqslant \cdots \leqslant x_{n-1} \leqslant 1$ such that

$$f([0, x_1)) = f([x_1, x_2)) = \cdots = f([x_{n-1}, 1]) .$$

In other words: existence of envy-free division when simultaneously

- the players have the same preferences.
- the preferences are obtained by "weighing" the pieces.

In $\displaystyle\bigcap_{p \text{ prime divisor of } n} \text{PPA-}p$?

Is it a hard problem?

Plan

Statement

subset S of $[n]$ is 2-stable = it does not contain adjacent elements (1 and n being also adjacent).

For $n \geqslant 2k$, the Schrijver graph $\mathrm{SG}(n,k)$ has the 2-stable k-subsets of $[n]$ as vertex set; its edges connect disjoint vertices.

Theorem Simonyi, Tardos, Vrećica 2009

In any proper coloring of $\mathrm{SG}(n,k)$, there is a vertex with at least $\lfloor n/2 \rfloor - k + 3$ colors in its closed neighborhood.

Remarks:

- $\lceil n/2 \rceil - k + 2$ results from standard techniques
- formulated in general in terms of the local chromatic number

Proof

As (almost) always for colorings of Schrijver graphs, the proof is topological.

Uses the following result.

Theorem Ščepin 1974

For every map $f\colon S^{2t} \to \mathsf{K}$ to some simplicial complex K of dimension t, there exists $x \in S^{2t}$ so that $f(-x) = f(x)$.

Simpler proof by Vrećica (2018).

Might show that finding a "large" colorful neighborhood in a Schrijver graph belongs to PPA.
(Model: colors given by a polynomial time function)

Plan

Summary

- Independent transversal: in PLS? in PPAD? Hard problem?
- Győri–Lovász theorem: in PPAD? PLS-complete?
- Kernels in clique-acyclic perfect digraphs: in PPAD? Hard problem?
- Rainbow fractional matchings: in PPAD? Hard problem?
- Envy-free cake-cutting, mixed case: in PPA-p?
- Equipartition of a segment: in $\bigcap_p$ PPA-p? Hard problem?
- Large colorful neighborhood in Schrijver graphs: PPA? Hard problem?

Plan

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Thank you