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Comparison of Formulations for the Inventory Routing Problem

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Outline

- Introduction to the IRP
- Aggregated formulations

  Equivalence between compact and exponential-size formulations: Polyhedral projection

  - Capacity constraints
  - Connectivity constraints
  - Multi-star inequalities
- Comparison with disaggregated formulations
- Computational analysis
- Conclusions

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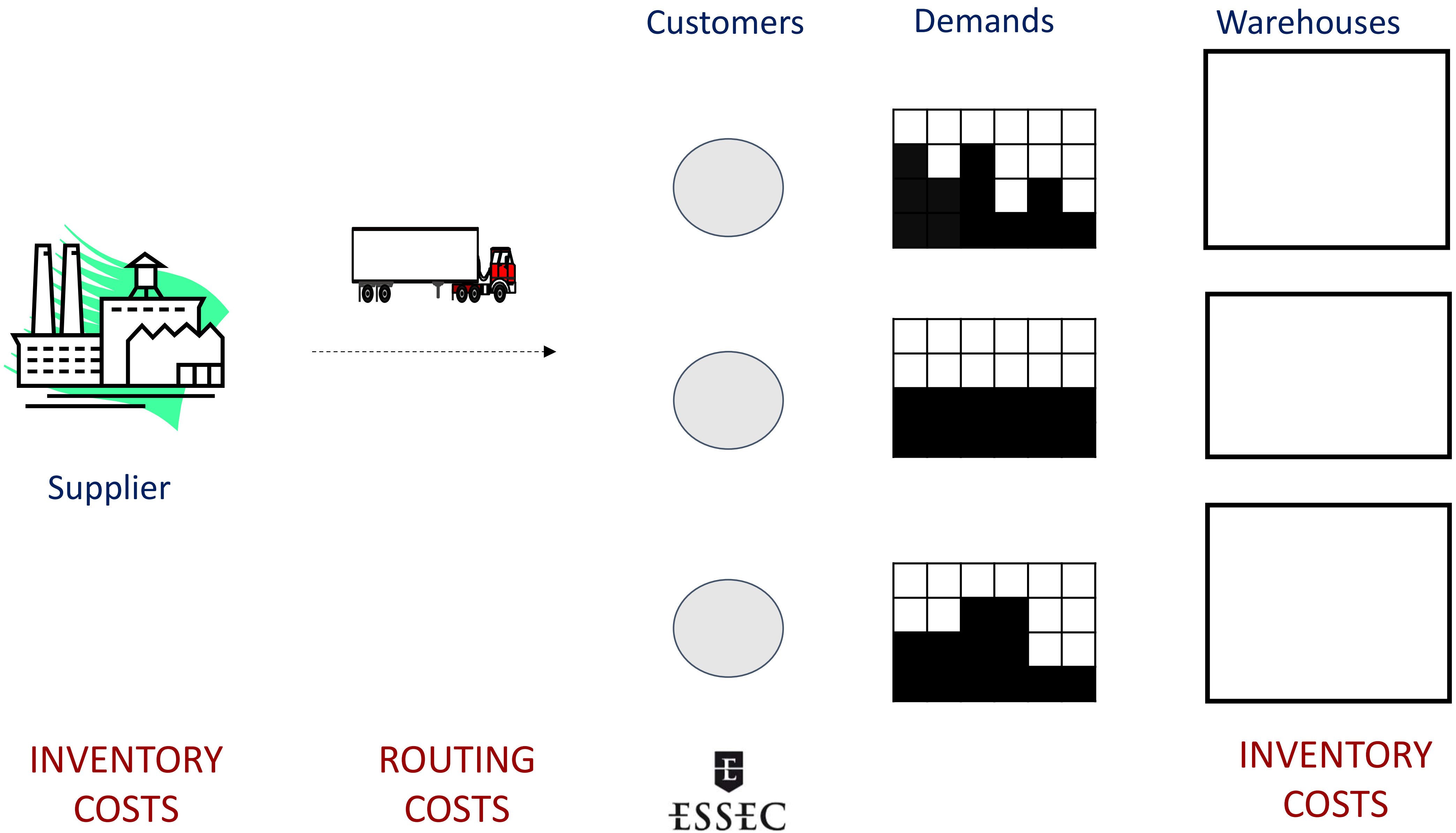
Inventory Routing Problem (IRP)



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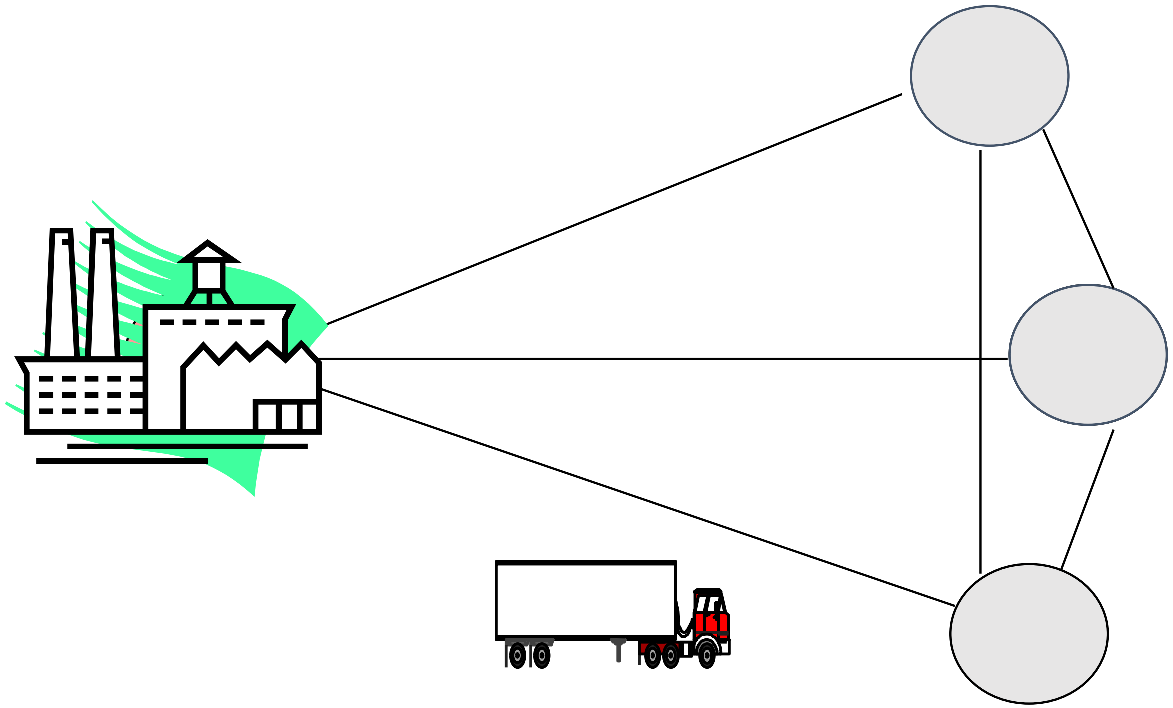
Inventory Routing Problem (IRP)



Find

a distribution plan over a **planning horizon**

that minimizes

routing costs

and inventory holding costs

Literature: exact approaches

Branch-and-cut algorithms

- Coelho, Cordeau, Laporte (2012), C&OR, (2012), TRC
- Adulyasak, Cordeau, Jans (2012), IJC
- Coelho, Laporte (2013), C&OR, (2013) IJPR, (2014) IJPE
- Archetti, Bianchessi, Irnich, Speranza, ITOR (2014)
- Avella, Boccia, Wolsey (2015), Networks, (2018), TS
- Manousakis, Repoussis, Zachariadis, Tarantilis (2021) EJOR

Branch-and-price algorithms

- Desaulniers, Rakke, Coelho, (2015) TS

Literature: aggregated vs. disaggregated formulations

Aggregated formulations: no vehicle index

- Adulyasak, Cordeau, Jans (2012), IJC
- Archetti, Bianchessi, Irnich, Speranza, ITOR (2014)
- Avella, Boccia, Wolsey (2015), Networks, (2018), TS
- Manousakis, Repoussis, Zachariadis, Tarantilis (2021) EJOR

Disaggregated formulations: vehicle index

- Coelho, Cordeau, Laporte (2012), C&OR, (2012), TRC
- Coelho, Laporte (2013), C&OR, (2013) IJPR, (2014) IJPE
- Archetti, Bianchessi, Irnich, Speranza, ITOR (2014)

IRP formal definition

- **Directed** complete graph G=(N,A), where N= 0 (supplier, depot)∪ N′ (customers)
- *n* customers
- *T* set of time periods {1,…,*H*}, *H* horizon
- Fleet *K* of *m* homogeneous vehicles with capacity *Q*
- Production rate at the supplier r_{0t}
- Daily demand at the customers r_{it}
- Maximum inventory level at customers U_i
- Initial inventory level I_{i0}
- **Split deliveries are not allowed**
- Routing cost c_{ij} that satisfy triangle inequality
- Inventory cost at customers and supplier h_i

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IRP formal definition

Find the distribution plan:

Delivery schedule

+

Routing

Minimizing the total cost: routing + inventory

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Polyhedral projection and equivalent formulations

Given a MIP formulation *A*, by P_A we denote the polyhedron of its LP relaxation

in which discrete variables are replaced by continuous ones. Given a formulation *A*

in the extended space of (*x; g*) variables, its natural projection into the space of *x*

variables, is

$$Proj_x(P_A) = \{x \mid (x, g) \in P_A\}$$

Given two MIP formulations, *A* and *B*, we say that *A* is at least as strong as *B* if for

any problem instance, the value of the LP-relaxation of the formulation *A* is at least

as good as the value of the LP-relaxation of the formulation B

Aggregated formulations

Variables

- I_i^t: continuous inventory variables
- Q_i^t: continuous quantity variables
- Z_i^t: binary visiting variables associated with customers
- Z_0^t: integer variables counting the number of routes performed at time t
- X_{ij}^t: binary routing variables

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Aggregated formulations

$$\min \quad \sum_{t\in T} h_0 I_{0t} + \sum_{i\in N'}\sum_{t\in T} h_i I_{it} + \sum_{(i,j)\in A}\sum_{t\in T} c_{ij} X_{ij}^t$$

Objective function

$$\text{s.t.} \quad I_{0t} = I_{0,t-1} + r_{0t} - \sum_{i\in N'} Q_i^t \qquad t\in T$$

$$I_{it} = I_{i,t-1} - r_{it} + Q_i^t \qquad i\in N',\ t\in T$$

$$(A) \quad Q_i^t \le U_i - I_{it-1} \qquad i\in N',\ t\in T$$

$$Q_i^t \le C_i^t Z_i^t \qquad i\in N',\ t\in T$$

Inventory coinstraints where

$$C_i^t := \min\{U_i, Q, \sum_{t'=t}^{H} r_{it'}\}$$

$$X^t(\delta^+(i)) = X^t(\delta^-(i)) \qquad i\in N,\ t\in T$$

$$X^t(\delta^-(i)) = Z_i^t \qquad i\in N,\ t\in T$$

Routing coinstraints

$$Z_i^t \in \{0,1\} \qquad i\in N',\ t\in T$$

$$Z_0^t \in \{0,1,\ldots,|K|\} \qquad t\in T$$

$$X_{ij}^t \in \{0,1\} \qquad \{i,j\}\in A,\ t\in T$$

$$Q_i^t \ge 0, I_{it} \ge 0 \qquad i\in N,\ t\in T$$

Variables domain

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Aggregated formulations

Formulation A may give infeasible solutions because of:

- Capacity constraints
- Connectivity constraints

Capacity constraints

Compact formulation: Load-based formulation (LOAD)

- l_{ij}^{t}: continuous load variables measuring the load of the vehicle while traversing the arc (*i; j*) in day *t*

$$\ell^t(\delta^-(i)) - \ell^t(\delta^+(i)) = \begin{cases} Q_i^t & \text{if } i \neq 0, \\ -\sum_{i \in N'} Q_i^t & \text{if } i = 0. \end{cases} \qquad i \in N, t \in T \qquad (2a)$$

$$0 \leq \ell_{ij}^t \leq QX_{ij}^t \qquad (i,j) \in A, t \in T \qquad (2b)$$

Constraints (2) guarantee capacity and connectivity constraints

Formulation (A) + (2) is a valid IRP formulation (LOAD)

Capacity constraints

Exponential formulation: Fractional Capacity Cuts (FCC)

$$X^t(\delta^-(S)) \geq \frac{1}{Q} Q^t(S) \quad S \subseteq N', \ t \in T$$

or

$$X^t(\delta^+(S)) \geq \frac{1}{Q} Q^t(S) \quad S \subseteq N', \ t \in T$$

FCC guarantee capacity and connectivity constraints

Formulation (A) + FCC is a valid IRP formulation (A+FCC)

Comparison between LOAD and A+FCC

Theorem

There is a one-to-one correspondence between solutions of the LP-relaxation of the LOAD and the solutions of the LP-relaxation of A+FCC:

$$Proj_{(Z,Q,X)}(P_{LOAD}) = P_{A+FCC}.$$

This result is in-line with what is known for the capacitated VRP (Gouveia, 1995 EJOR)

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Comparison between LOAD and A+FCC

- LOAD and A+FCC give the same value of LP relaxation
- LOAD is compact: does not require any dynamic separation
- No B&C needed for LOAD, contrary to A+FCC
- Commercial solvers typically behave better on complete compact formulations:
  - automated cuts are disabled when using callbacks
  - Generic heuristics work better on complete formulations

Strengthened Load-Based Formulation and Multi-Star Inequalities for the IRP

Lemma

When input parameters ($Q; r; U$) take on integer values, then there exists

an optimal solution such that the values of the quantities Q^t_i are integer

Strengthened Load-Based Formulation (SLOAD)

The LOAD formulation can be strengthened by replacing constraints (2)

with the following ones:

$$X_{ij}^t \leq \ell_{ij}^t \leq (Q-1)X_{ij}^t \quad i,j \in N',\ t \in T \tag{14a}$$

$$\ell_{j0}^t = 0 \quad j \in N',\ t \in T \tag{14b}$$

$$X_{0j}^t \leq \ell_{0j}^t \leq QX_{0j}^t \quad j \in N',\ t \in T \tag{14c}$$

Formulation LOAD+(14) is called SLOAD

Multi-Star Inequalities for the IRP

Definition

Let us consider a set $S \in N'$ and $t \in T$. Then:

$$QX^t(\delta^-(S)) \geq Q^t(S) + X^t(S^c : S) + X^t(S : S^c)$$

are called IRP-Multi-Star inequalities (MS). They strenghten the FCC through the second and third term of the RHS

Comparison between SLOAD and A+MS

Theorem

There is a one-to-one correspondence between solutions of the LP-relaxation of the SLOAD and the solutions of the LP-relaxation of A+MS:

$$Proj_{(Z,Q,X)}(P_{SLOAD}) = P_{A+MS}.$$

Same observations as for LOAD and A+FCC…

Connectivity constraints

Connectivity is guaranteed through the formulations seen earlier

HOWEVER

One may add connectivity constraints that are not implied by the former capacity/MS constraints and, thus, may strengthen the value of the relaxation

Connectivity constraints

Compact formulation: Multi-Commodity Flow (MCF)

- f_{ij}^{tl}: continuous flow variables representing the path from the depot to customer *l* in day *t*

$$f^{tl}(\delta^{-}(i)) - f^{tl}(\delta^{+}(i)) = \begin{cases} Z_i^t & \text{if } i = l, \\ -Z_i^t & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases} \quad l \in N',\ i \in N,\ t \in T \quad (16a)$$

$$0 \leq f_{ij}^{tl} \leq X_{ij}^t \quad l \in N', (i,j) \in A,\ t \in T \quad (16b)$$

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Connectivity constraints

Exponential formulation: Generalized Subtour Elimination

Constraints (GSEC)

$$X^t(A(S)) \leq Z^t(S \setminus \{i\}) \quad S \subseteq N', |S| \geq 2, \; i \in S, \; t \in T.$$

Comparison between LOAD+MCF and A+FCC+GSEC

Theorem

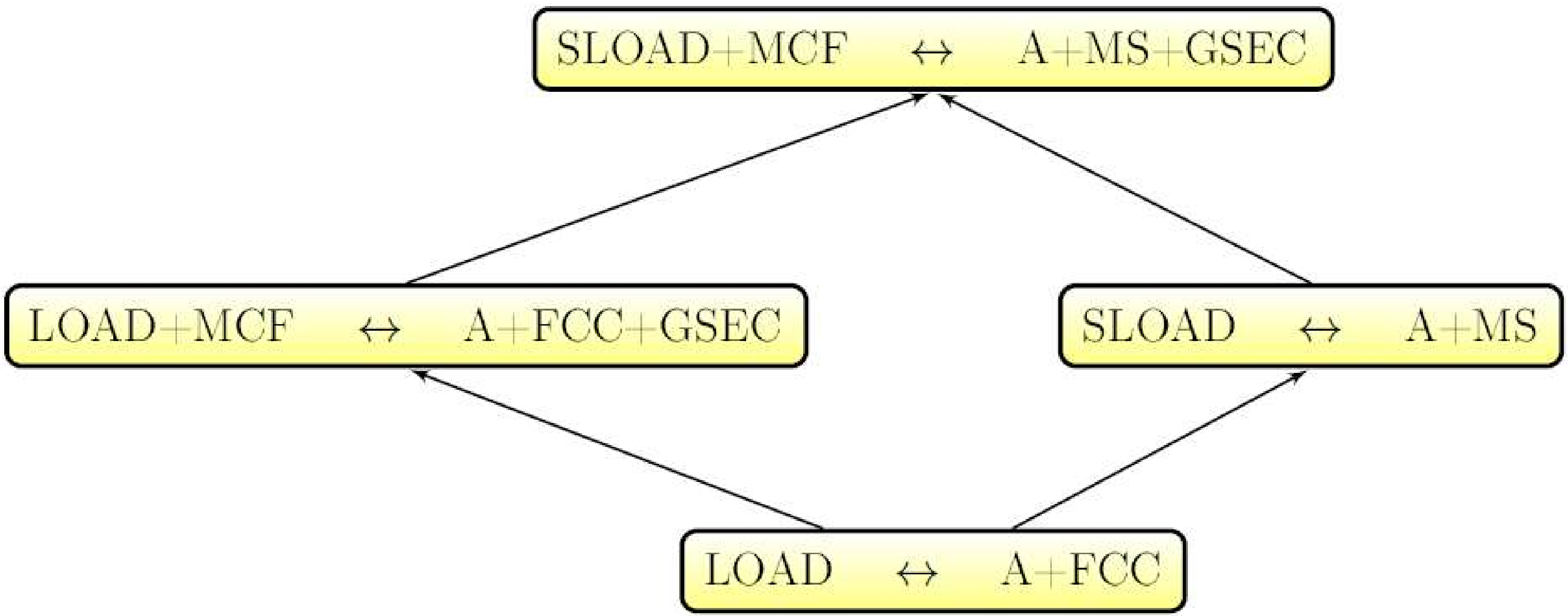
There is a one-to-one correspondence between solutions of the LP-relaxation of

the LOAD+MCF and the solutions of the LP-relaxation of A+FCC+GSEC:

$$Proj_{(X,Z,Q)}(P_{LOAD+MCF}) = P_{A+FCC+GSEC}.$$

Same observations as for LOAD and A+FCC…

Hierarchy of aggregated formulations



A ⟶ B: A is at least as strong as B in terms of linear relaxation

Disaggregated formulations

Variables

- I_i^t: continuous inventory variables
- q_i^{kt}: continuous quantity variables
- z_i^{kt}: binary visiting variables associated with customers and depot
- x_{ij}^{kt}: binary routing variables

Disaggregated formulations

Objective function

$$\min \quad \sum_{t\in T} h_0 I_{0t} + \sum_{i\in N'}\sum_{t\in T} h_i I_{it} + \sum_{k\in K}\sum_{(i,j)\in A}\sum_{t\in T} c_{ij} x_{ij}^{kt}$$

Inventory coinstraints where

$$\text{s.t.} \quad I_{0t} = I_{0,t-1} + r_{0t} - \sum_{k\in K}\sum_{i\in N'} q_{it}^k \qquad t \in T$$

$$I_{it} = I_{i,t-1} - r_{it} + \sum_{k\in K} q_{it}^k \qquad i \in N',\ t\in T$$

$$(D) \quad \sum_{k\in K} q_{it}^k \leq U_i - I_{it-1} \qquad i\in N',\ t\in T$$

$$0 \leq q_{it}^k \leq C_i^t z_i^{kt} \qquad i\in N',\ k\in K,\ t\in T$$

Capacity coinstraints

$$\sum_{i\in N'} q_{it}^k \leq Q z_0^{kt} \qquad k\in K,\ t\in T$$

No-split coinstraints

$$\sum_{k\in K} z_i^{kt} \leq 1 \qquad i\in N',\ t\in T$$

Routing coinstraints

$$x^{kt}(\delta^-(i)) = x^{kt}(\delta^+(i)) \qquad i\in N,\ k\in K,\ t\in T$$

$$x^{kt}(\delta^-(i)) = z_i^{kt} \qquad i\in N,\ k\in K,\ t\in T$$

Variables domain

$$z_i^{kt} \in \{0,1\} \qquad i\in N,\ k\in K,\ t\in T$$

$$x_{ij}^{kt} \in \{0,1\} \qquad (i,j)\in A,\ k\in K,\ t\in T$$

$$I_{it} \geq 0 \qquad i\in N,\ t\in T$$

Disaggregated formulations

Formulation D may give infeasible solutions because of:

- Connectivity constraints

Disaggregated connectivity constraints

Connectivity constraints may be imposed through

- Disaggregated GSECs (dGSECs)

$$x^{kt}(A(S)) \leq z^{kt}(S \setminus \{i\}) \quad S \subseteq N',\ i \in S,\ t \in T,\ k \in K$$

- Disaggregated FCC (dFCC)

$$x^{kt}(\delta^{-}(S)) \geq \frac{1}{Q} \sum_{i \in S} q_i^{kt} \qquad S \subseteq N',\ t \in T,\ k \in K$$

Strength of the disaggregated formulations

Theorem

There is a one-to-one correspondence between solutions of the LP-relaxation of

the D+FCC+GSEC and the solutions of the LP-relaxation of A+FCC+GSEC:

$$Proj_{(Z,Q,X)}(P_{D+FCC+GSEC}) = P_{A+FCC+GSEC}$$

Nothing is gained through disaggregation!

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Main conclusions from theoretical analysis

- Compact formulations are as strong as exponential ones

- Aggregated formulations are as strong as disaggregated formulations

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Computational analysis

- Aims at verifying the computational efficacy of the aggregated formulations

- Comparison with state-of-the-art approaches:

  - B&C algorithm from Coelho and Laporte (2014) IJPE - **CL**:
    - Disaggregated formulation
    - GSECs separated dynamically
    - Complete UNDIRECTED graph

  - B&P algorithm from Desaulniers, Rakke, Coelho, (2015) TS – **DRC**:
    - Set-partitioning formulation
    - Column generation with complex set of domination rules

Computational tests: instances

Benchmark IRP instances:

- $n = 5 - 50$
- $H = 3, 6$
- $m = 2 - 5$
- Low & High inv. Costs

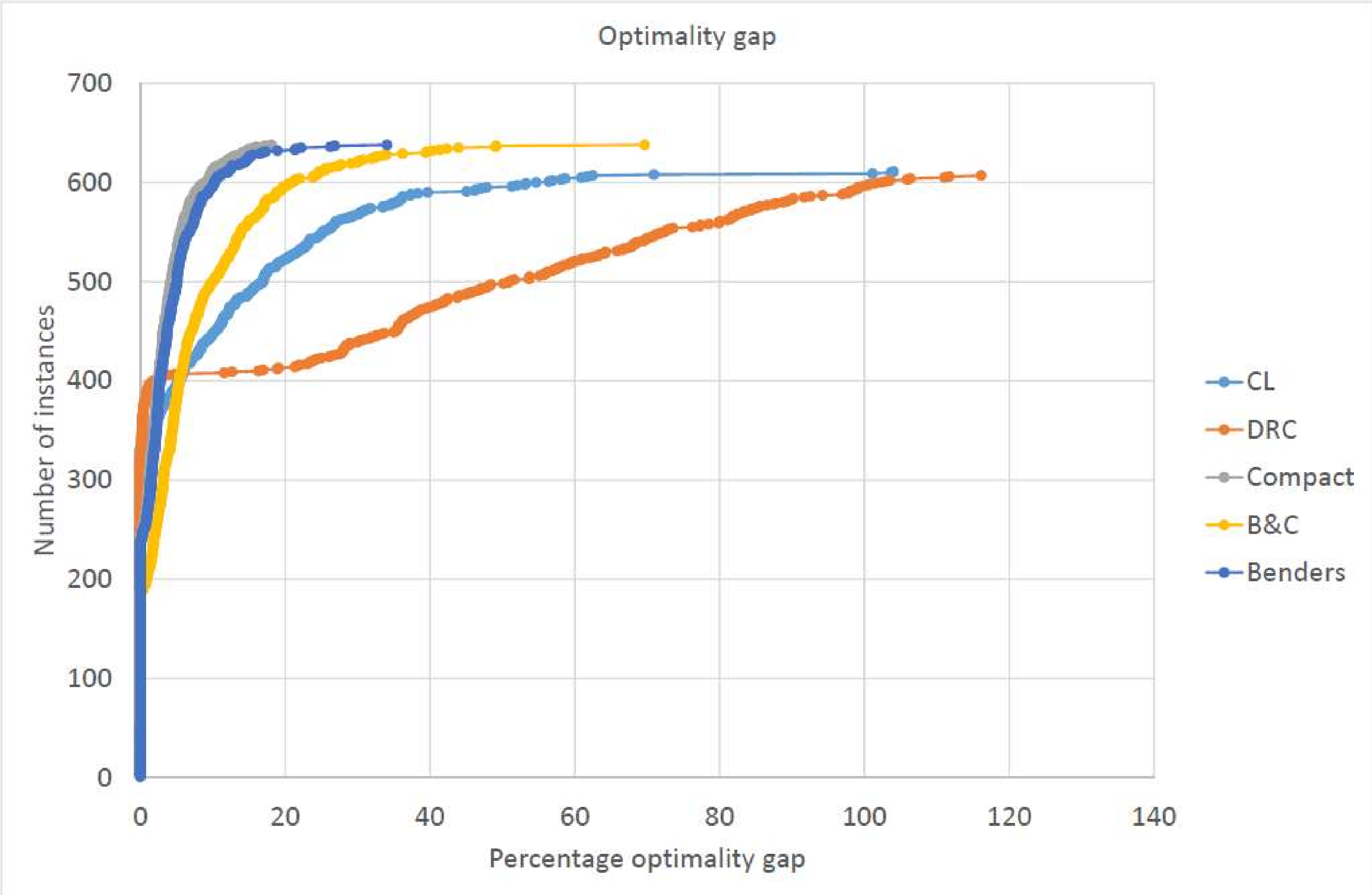
640 instances

Solution approaches

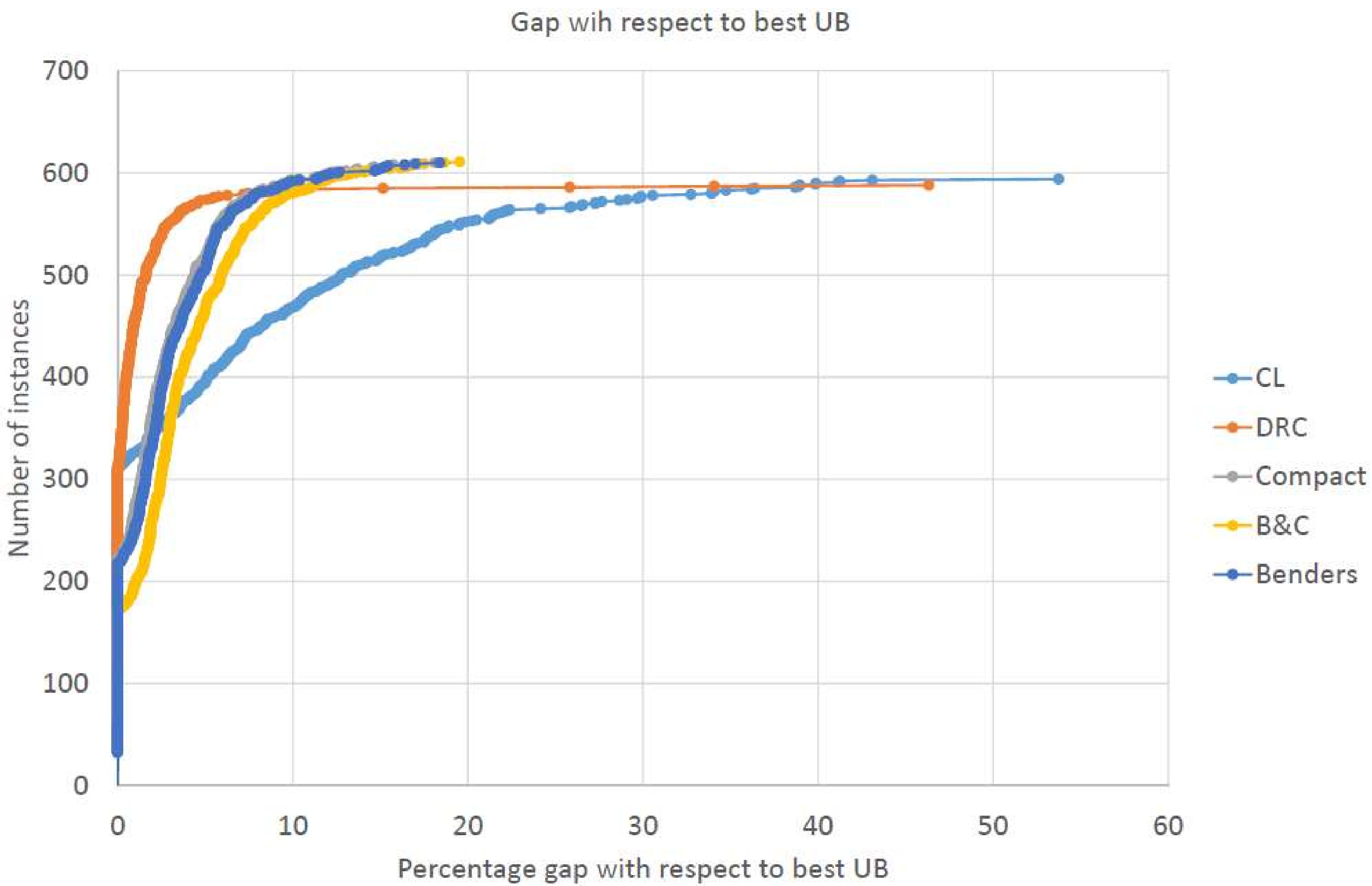
- **Compact**: SLOAD
- **B&C**: SLOAD + GSECs separated on the fly through the classical min-cut algorithm
- **Benders**: SLOAD + MCF inserted through Cplex annotated Benders to avoid the computational burden of introducing f variables. Subproblems are separated by t and l as they are fully independent

Time limit: 2 hours

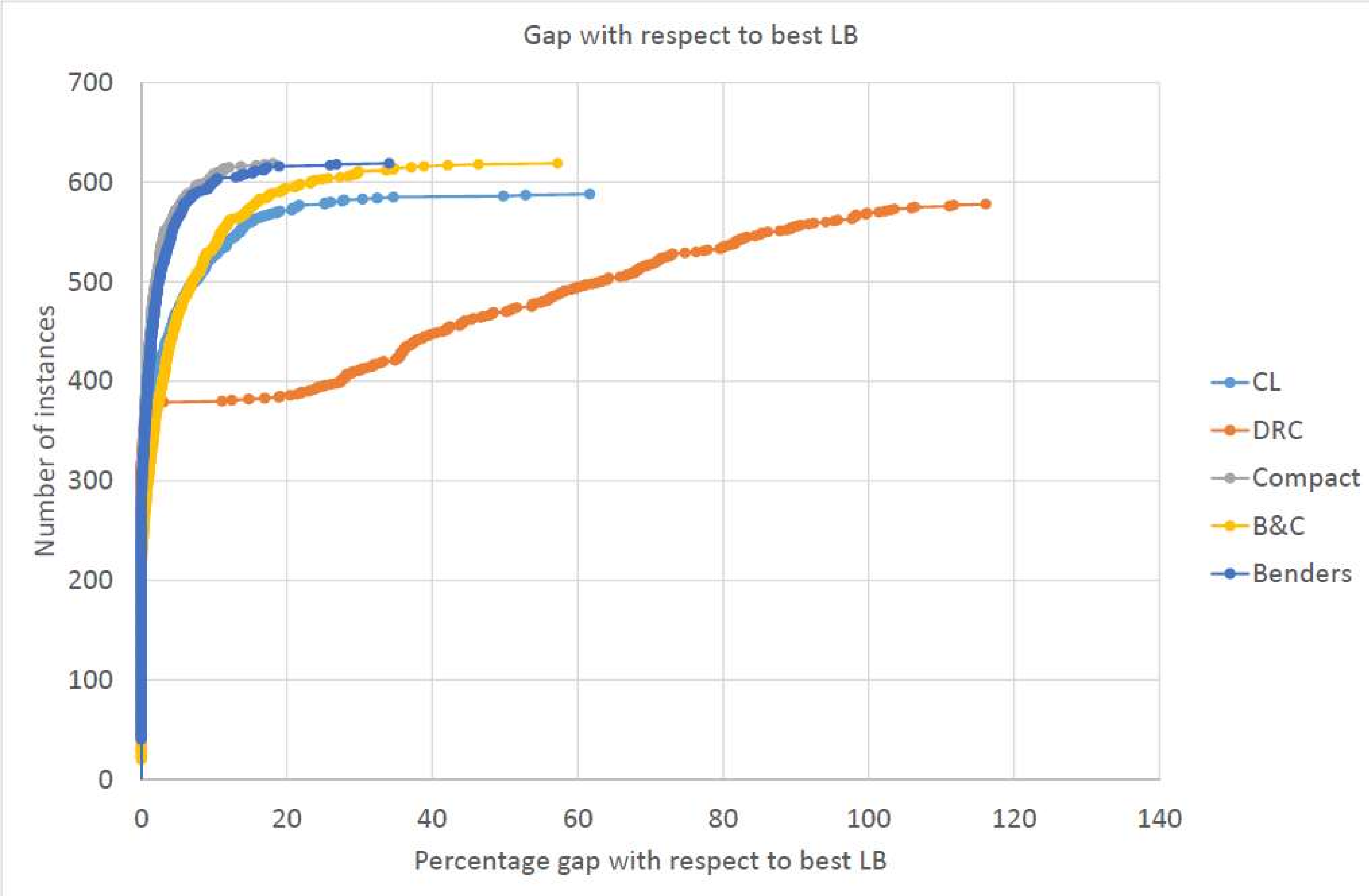
Results: Optimality gap at termination



Results: Gap between lower bound at termination and best upper bound



Results: Gap between upper bound at termination and best lower bound



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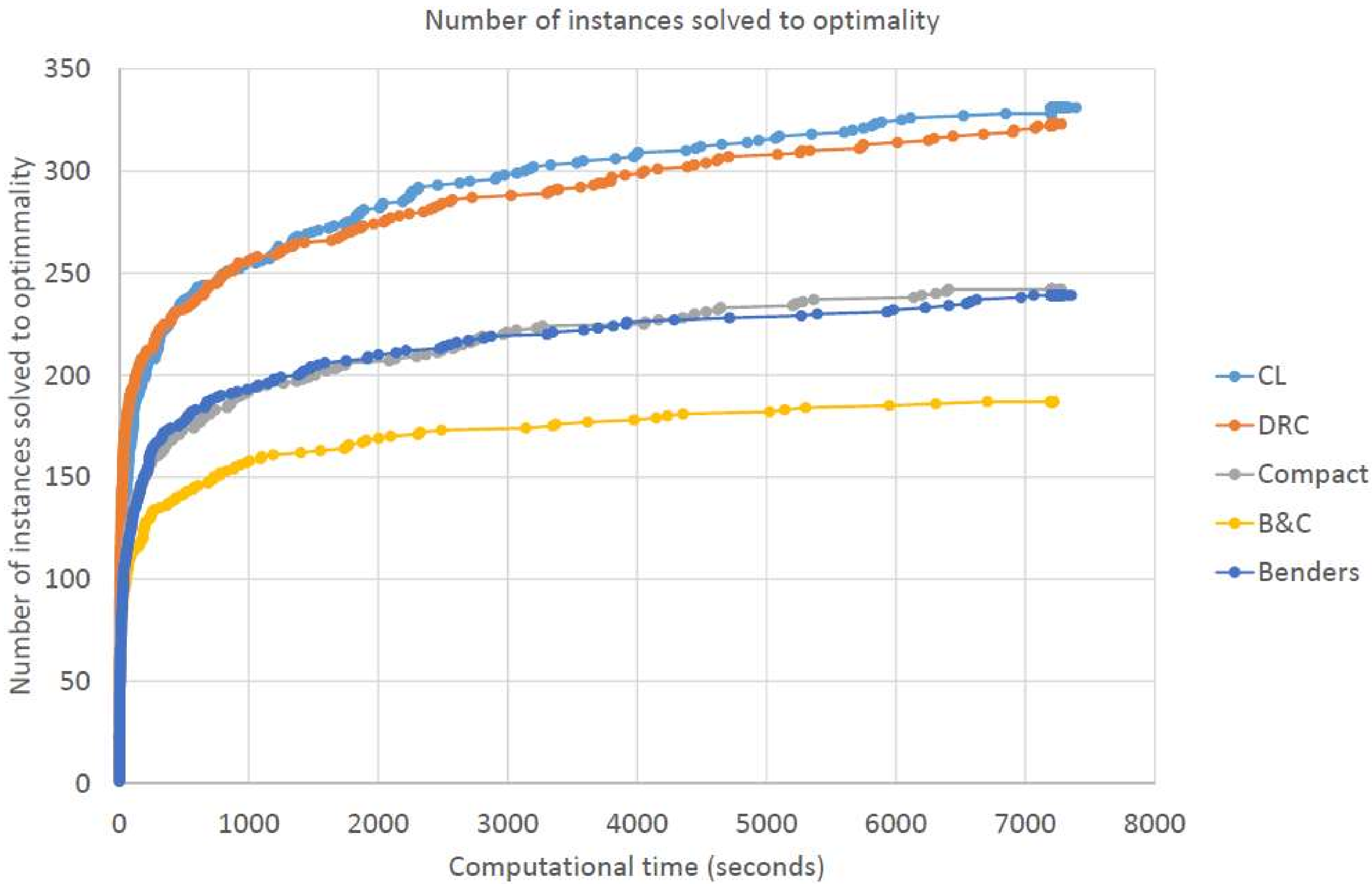
Results: Number of instances solved to optimality vs. computing time



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Remarks: Comparison among approaches with aggregated formulations

- Compact behaves better than B&C and Benders

- BC is worse than Compact and Benders

Remarks: Comparison with CL and DRC

- Aggregated formulations provide good UB and LB:
  - Optimality gap remains below 20% for Compact, below 40% for Benders and below 70% for B&C while it goes up to more than 100% for CL and DRC
  - Gap between LB and best UB is below 20% for aggregated approaches while CL and DRC go above 40% and 50%
  - Gap between UB and best LB is at most 20% for Compact, 35% for Benders and 60% for B&C while CL and DRC go above 60% and 120%
- However, they solve less instances to optimality (2 hour limit)

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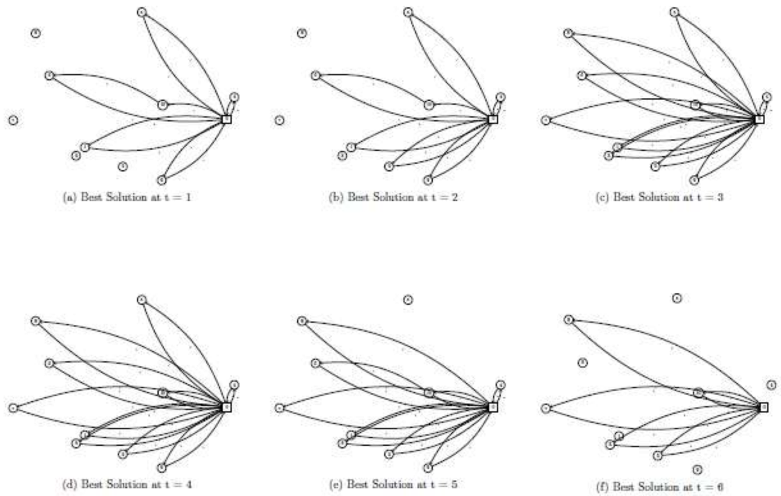
Solvers' statistics: number of nodes

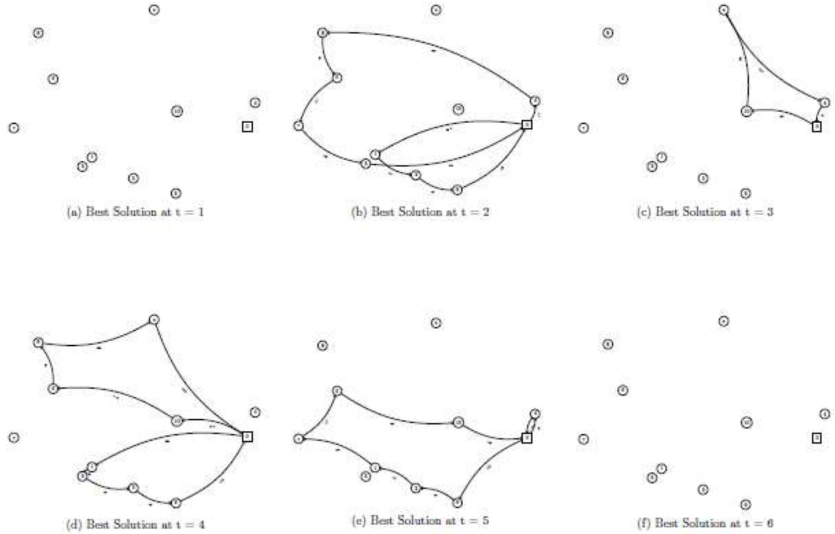
	Compact	B&C	Benders
Av. *Low*	94939	59063	68865
Av. *High*	75823	55178	57291
Total av.	85381	57121	63078

The number of nodes in DRC is nearly one order of magnitude lower than Compact

Remark about LOAD relaxation

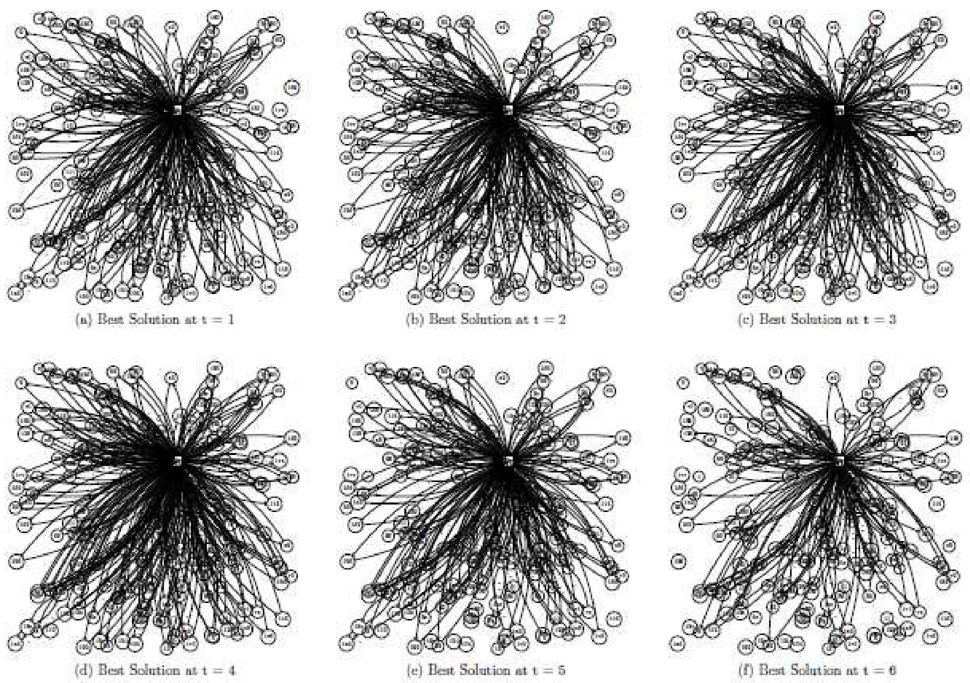
Archetti, Huerta-Munoz, Guastaroba, Speranza, A Kernel Search Heuristic for the Multi-Vehicle Inventory Routing Problem, ITOR 28, 2021

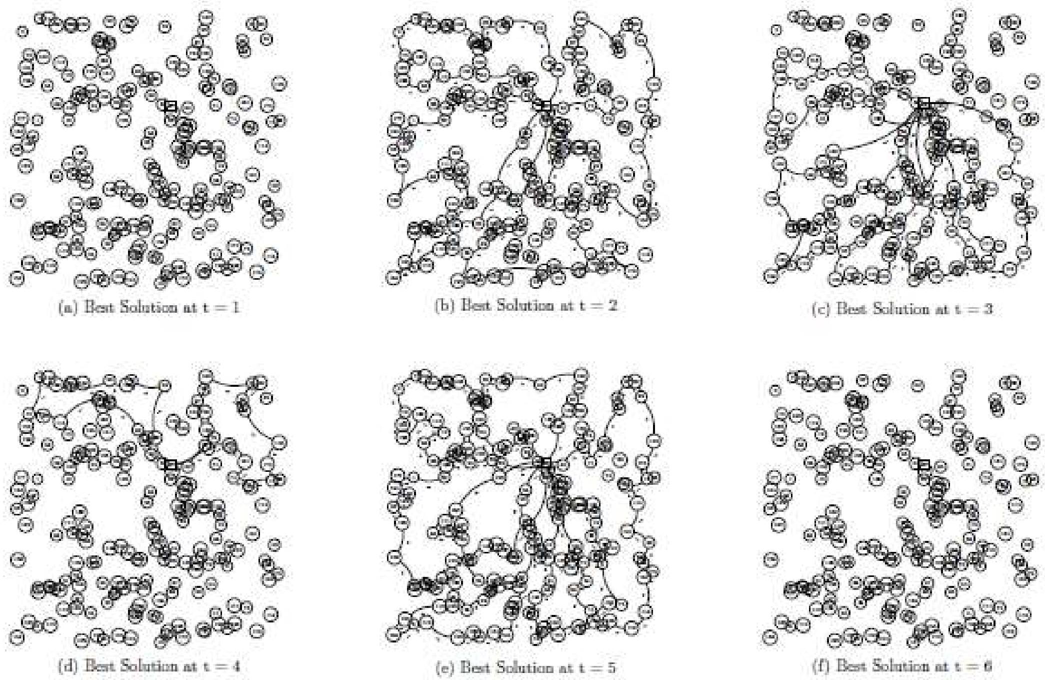




Linear relaxation and optimal integer solution for an instance with 10 customers

Remark about LOAD relaxation





Linear relaxation and best known solution for an instance with 200 customers

Conclusions

- Compact behaves better than B&C and Benders
- BC is worse than Compact and Benders
- Aggregated formulations provide good UB and LB
- However, they solve less instances to optimality (2 hour limit)

Future directions

Study the link between one-commodity and two-commodity formulations

(Manousakis, Repoussis, Zachariadis, Tarantilis (2021) EJOR)

THANK YOU FOR YOUR ATTENTION

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