New route formulations for the Split-Delivery VRP

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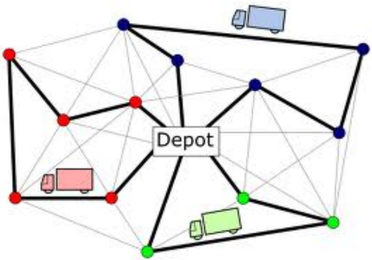
From standard CVRP to SDVRP's

Classic Capacitated Vehicle Routing Problem - CVRP

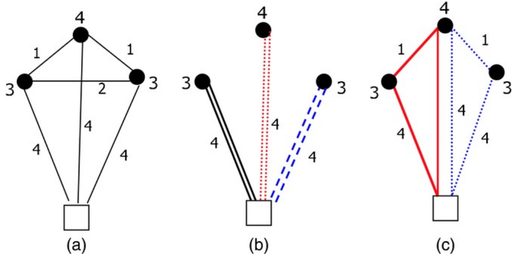
- Objective: minimise routing costs.



Split delivery variants – SDVRP's

- The single visit requirement for customers is relaxed.
- Each client can now be visited by one or more vehicles.

Practical motivation



Instance (a) with $Q = 5$. The cost is 24 (with 3 vehicles) for the CVRP (b)

and 18 (with 2 vehicles) for the SDVRP. Source: [Archetti and Speranza, 2012].

- Routing savings can reach up to 50% [Archetti et al., 2006].

Research motivation

CG-based approaches and BCP algorithms:

Feillet, Dejax, Gendreau and Gueguen (2006)
Jin, Liu and Eksioglu (2008)
Moreno, De Aragão and Uchoa (2010)
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Munari and Savelsbergh (2020)

BC algorithms (current state-of-the-art):

Archetti, Bianchessi and Speranza (2014)
Ozbaygin, Karasan and Yaman (2018)
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The SDVRP has a similar structure to the IRP!

Current BCP algorithms for the SDVRP's

- Based on extreme delivery patterns [Desaulniers, 2010]
- Pricing problem is harder than the standard RCSPP
- To use the standard RCSPP solver [Sadykov et al., 2021], we need to discretize delivery quantities
- Are there route formulations which allow us to use the standard RCSPP solver without full discretization?

Base formulation for SDVRP's

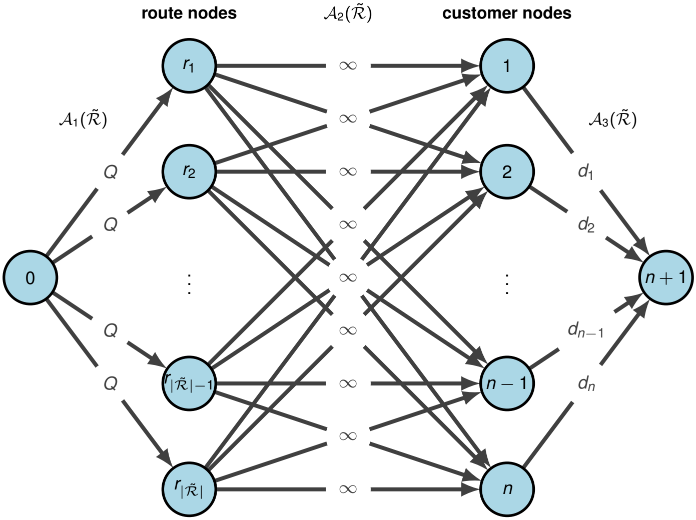
- $\mathcal{C}$ — set of customers
- $\mathcal{R}$ — set of elementary [and time-feasible] routes.
- c^r — cost of route $r \in \mathcal{R}$
- $h_{rS} = 1$ iff route $r \in \mathcal{R}$ enters subset $S \subseteq \mathcal{C}$ of customers
- θ_r — number of vehicles which follow route $r \in \mathcal{R}$ (variable)

$$
\begin{aligned}
\text{(F0):} \quad \text{Min} \quad & \sum_{r \in \mathcal{R}} c^r \theta_r, \\
\text{s.t.} \quad & \sum_{r \in \mathcal{R}} h_{rS} \theta_r \geq \left\lceil \sum_{i \in S} d_i / Q \right\rceil, && \forall S \subseteq \mathcal{C}, \\
& \theta_r \in \mathbb{Z}_+, && \forall r \in \mathcal{R}.
\end{aligned}
$$

- Constraints are strong k-path inequalities [Baldacci et al., 2008, Archetti et al., 2011].
- No information about delivery quantities in route variables!

Flow graph $\mathcal{F}(\tilde{\mathcal{R}})$ to show correctness of (F0)

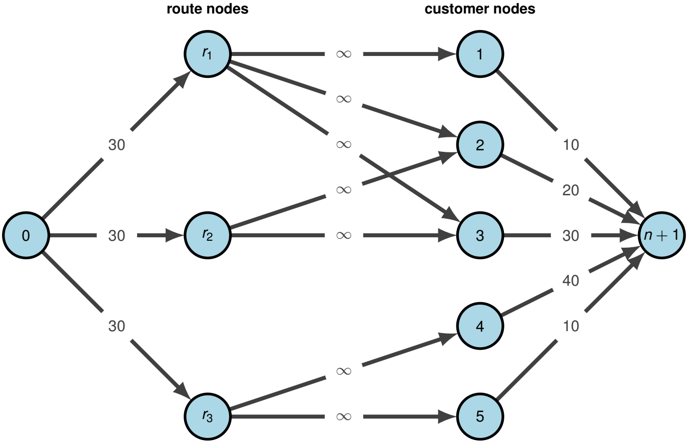
$\tilde{\mathcal{R}}$ is the set of routes in the solution of (F0)



An example of flow graph $\mathcal{F}(\tilde{\mathcal{R}})$

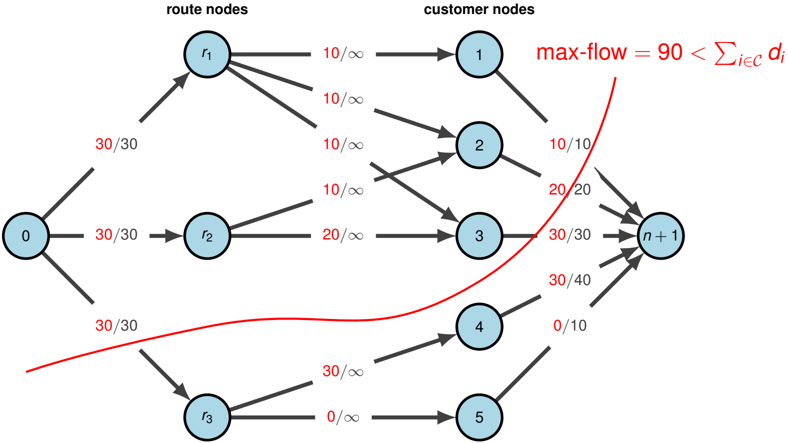
Customers $\mathcal{C} = \{1, 2, 3, 4, 5\}$ with demands $d = \{10, 20, 30, 40, 10\}$, and vehicle capacity $Q = 30$.

$\tilde{\mathcal{R}} = \Big\{ r_1 = \{0, 1, 2, 3, 6\}, r_2 = \{0, 2, 3, 6\}, r_3 = \{0, 4, 5, 6\} \Big\}$



Checking feasibility with $\mathcal{F}(\tilde{\mathcal{R}})$

The max-flow value in $\mathcal{F}(\tilde{\mathcal{R}})$ tells us if $\tilde{\mathcal{R}}$ is a feasible solution.

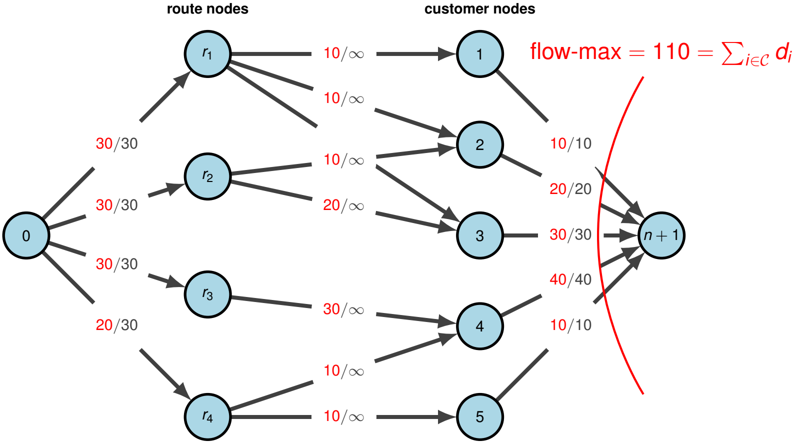


$\sum_{r\in\tilde{\mathcal{R}}} h_{r,\{4,5\}}\theta_r$ (1) $<$ $\lceil\sum_{i\in\{4,5\}} d_i/Q\rceil$ ($\lceil 50/30\rceil = 2$)

$\Rightarrow$ strong k-path inequality for $S = \{4, 5\}$ is violated

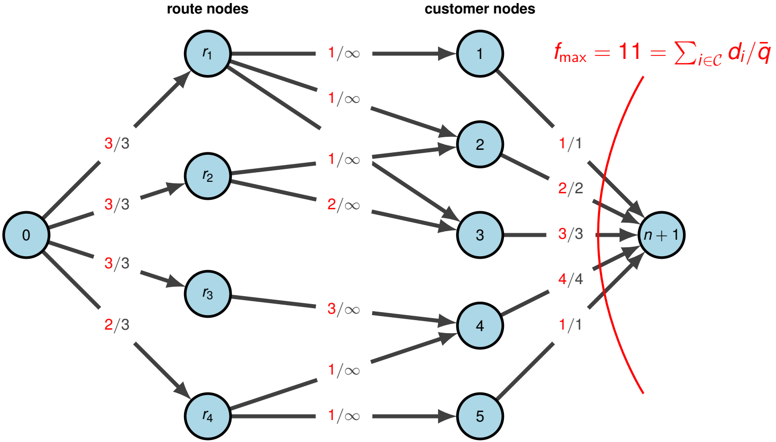
Checking feasibility with $\mathcal{F}(\tilde{\mathcal{R}})$ (II)

$$\tilde{\mathcal{R}}^* = \Big\{ r_1 = \{0,1,2,3,6\}, r_2 = \{0,2,3,6\}, r_3 = \{0,4,6\}, r_4 = \{0,4,5,6\} \Big\}$$



A dominance rule for optimal solutions

Divide arc capacities in $\mathcal{F}(\tilde{\mathcal{R}})$ by $\bar{q} = gcd(Q, d_1, d_2, \ldots, d_n)$.



Dominance rule: *There exists an optimal solution in which all delivery quantities in all routes are multiples of* $\bar{q}$

Strengthened formulation (F2)

- $\mathcal{R}'$ — set of all resource-feasible routes (but not necessarily elementary)
- $D_i = \{\bar{q}, 2\bar{q}, \ldots, d_i\}$ — possible delivery quantities to $i \in \mathcal{C}$.
- $b^r_{i\mathrm{F}} = b^r_{i,d_i}$ — # of times $r \in \mathcal{R}'$ delivers full demand to $i \in \mathcal{C}$.
- $b^r_{i\mathrm{P}} = \sum_{q \in D_i \setminus \{d_i\}} b^r_{iq}$ — # of times $r \in \mathcal{R}'$ delivers partial demand to i.

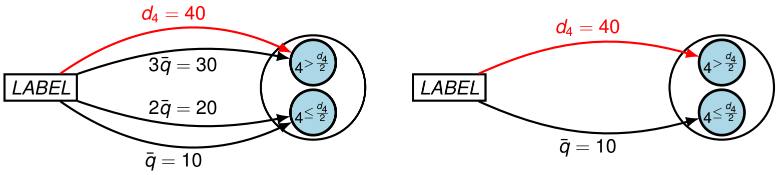
(F2) : Objective and all constraints in (F0)

$$\sum_{r \in \mathcal{R}'} (2b^r_{i\mathrm{F}} + b^r_{i\mathrm{P}})\theta_r \geq 2, \quad \forall i \in \mathcal{C}. \qquad (*)$$

(*) is a special case of strong minimum number of vehicles (SVM) constraints from [Archetti et al., 2011].

Pricing problem for formulation (F2)

Example: $i = 4$, $d_4 = 40$, $\bar{q} = 10$.



- Arcs incoming to nodes i with delivery $q \notin \{\bar{q}, d_i\}$ can be removed without compromising correctness
- Their removal does not weaken formulation (F2)

A family of formulations (FK)

A valid inequality for a customer $i \in \mathcal{C}$

$$\sum_{r\in\mathcal{R}'}\sum_{q\in D_i}(qb^r_{iq})\theta_r \geq d_i, \qquad (*)$$

where b^r_{iq} is the # of times $r \in \mathcal{R}'$ visits $i \in \mathcal{C}$ delivering $q \in D_i$.

Given $K < d_i/\bar{q}$, after Chvátal-Gomory rounding with multiplier $\frac{K-1}{d_i-\epsilon}$:

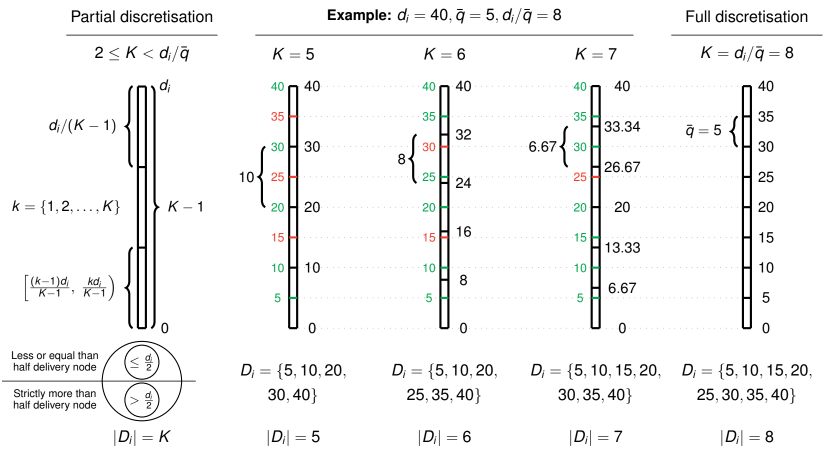
$$\sum_{r\in\mathcal{R}'}\sum_{q\in D_i}\sum_{k=1}^{K}(b^r_{iq}g^k_{iq}k) \geq K, \qquad (**)$$

where $g^k_{iq} = 1$ iff $\frac{(k-1)d_i}{K-1} \leq q < \frac{kd_i}{K-1}$.

$$\begin{aligned} (\text{F}K): \quad & \text{Objective and all constraints in (F0)} \\ & \text{Inequalities } (*) \quad \forall i \in \mathcal{C}: \; K \geq d_i/\bar{q} \\ & \text{Inequalities } (**) \quad \forall i \in \mathcal{C}: \; K < d_i/\bar{q} \end{aligned}$$

From partial to full discretisation: illustration

- Number of incoming arcs for vertices $i \in \mathcal{C}$ in the pricing for (FK) is at most K.
- Full discretisation formulation (F$K_{\max}$), $K_{\max} = \max_{i \in \mathcal{C}} \left\{ \frac{d_i}{\bar{q}} \right\}$.



Valid inequalities

x_{ij}^r — # of times $r \in \mathcal{R}'$ follows arc $(i,j) \in \mathcal{A}$, $i,j \in \mathcal{C} \cap \{0\}$.

- Rounded capacity inequalities:

$$\sum_{r \in \mathcal{R}} \sum_{\substack{(i,j) \in \mathcal{A}: \\ |\{i,j\} \cap S| = 1}} x_{ij}^r \theta_r \geq 2 \left\lceil \sum_{i \in S} d_i / Q \right\rceil, \quad \forall S \subseteq \mathcal{C}.$$

- 3-row subset-row packing inequalities:

$$\sum_{r \in \mathcal{R}} \left\lfloor \sum_{i \in S} \sum_{\substack{q \in D_i: \\ q > d_i/2}} \frac{1}{2} b_{iq}^r \right\rfloor \theta_r \leq 1, \quad \forall S \subseteq \mathcal{C},\ |S| = 3.$$

Valid inequalities (II)

- 3-row subset-row covering inequalities:

$$\sum_{r \in \mathcal{R}} \left\lfloor \sum_{i \in S} \sum_{\substack{q \in D_i: \\ q>0}} \frac{1}{2} b_{iq}^r \right\rfloor \theta_r \geq 2, \quad \forall S \subseteq \mathcal{C},\ |S| = 3.$$

- Limited memory technique ([Pecin et al., 2017]) is used for all non-robust cuts.

Implementation

- C++ libraries BaPCod [Sadykov and Vanderbeck, 2021] and VRPSolver extension [Pessoa et al., 2020] are used to leverage all the latest advances on exact solution of the classic CVRP
- VRPSolver is extended with
  - separation procedures for strong k-path inequalities
  - covering sets (to support limited-memory Chvátal-Gomory rank-1 covering cuts and strong k-path inequalities in the pricing)
- Branching on arcs and Ryan-and-Foster branching

Computational evaluation

Instance sets

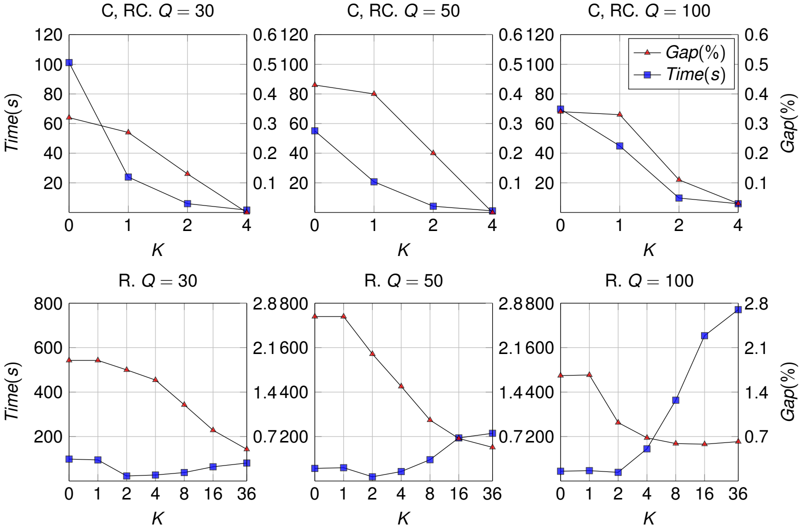
- **SDVRPTW** – 504 test instances, derived from 56 classic Solomon's VRPTW instances, having $n = \{25, 50, 100\}$ and $Q = \{30, 50, 100\}$.
- **SDVRP** – 352 test instances, derived from 88 instances (S, SD, eil, p), limiting, or not, the size of the fleet (LF/UF) and rounding, or not, distances (LF-r/UF-r).

Initial upper bounds

- We use an ILS-based matheuristic proposed by [Alvarez and Munari, 2022] to generate initial upper bounds.

Comparison of formulations (*FK*)

Root node results for all SDVRPTW instances with $n = 50$.



Comparison with the state-of-the-art on the SDVRPTW

n	Benchmark run – 3600s				Long run – 18000s		
	$(FK_{\max})$	MS22	BI19	A11	$(F2)$	$(FK_{\max})$	Best$(F2,\ FK_{\max})$
25	168	168	168	168	168	168	168 (0)
50	152 (27)	123	104	86	136	168	168 (40)
100	54 (48)	4	5	8	24	55	56 (50)
$\{50,\ 100\}$	206	127	109	94	160	223	224 (90)
$\{25,\ 50,\ 100\}$	374 (75)	295	277	262	328	391	392 (90)
OI average Gap (%)	1.66	-	-	-	3.02	1.56	1.57

MS22: Munari and Savelsbergh (2022)

BI19: Bianchessi and Irnich (2019)

A11: Archetti et al. (2011)

- Formulation $(FK_{\max})$ finds 374 optimal solutions, 75 for the first time, within one hour benchmark tests.
- Formulations $(F2)$ and $(FK_{\max})$ all together find 392 optimal solutions, 90 for the first time, within five hours.

Comparison with the state-of-the-art on the SDVRP

Formulation (FK), $K = \min(K_{\max}, 10)$

Tests	Model or reference – test set size	Opt	Opt*	LB*
Benchmark run – 7200s	*FK* (MH) – 352	94 (88†)	10 (6†)	121 (53†)
	Munari and Savelsbergh (2022) – 224†	85	-	-
	Gouveia et al. (2021) – 352	106	-	-
Long run – 18000s	*FK* (MH) – 352	112 (106†)	14 (10†)	130 (53†)
	FK (BKS) – 352	121 (115†)	19 (15†)	134 (53†)
	Best of long runs – 352	123 (117†)	20 (16†)	136 (54†)

† number of corresponding instances in the reduced test set considered in Munari and Savelsbergh (2022).

- Formulation (*FK*) finds 94 (88) optimal solutions, 10 (6) for the first time, within two hours benchmark tests.
- Our best results overall account for 123 (117) optimal solutions, 20 (16) for the first time, within five hours.

Conclusions

- A new family of partially discretised route formulations (FK) for SDVRP's.
- A new dominance rule ($\bar{q}$) for optimal SDVRP's solutions.
- Experimentally (FK) becomes stronger with $K \uparrow$
- BCP algorithm is the new state-of-the-art for the SDVRPTW

Perspectives

- Our BCP algorithm can be easily extended to other variants such as multiple depots [Gouveia et al., 2021], heterogeneous fleet [Belfiore and Yoshizaki, 2009], using the generic VRPSolver model.
- Further strengthening of formulation (FK_{max}) requires a generalized RCSPP solver fo the pricing
- We are bad for at finding good primal solutions!
- Extension to inventory and/or production routing problems?

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