Hybrid High-Order methods for finite deformations of hyperelastic materials

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ESMC 10th, Bologna, 03.07.18

Context

- Nonlinear hyperelastic problem
  - measure of the deformations (geometric nonlinearity)
  - stress-strain constitutive relation (material nonlinearity)
- Presence of volumetric-locking with primal H^1-conforming formulation in the incompressible limit $\lambda \to +\infty$ ($\nu \simeq 0.5$)
- An alternative : using mixed methods but more unknowns, more expensive to build, saddle-point problem to solve ...

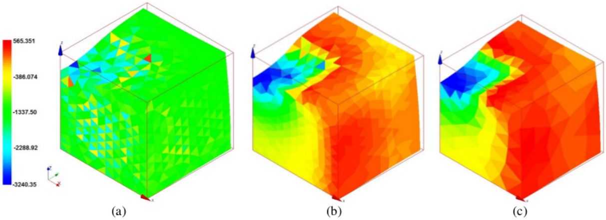


Figure 1 – Trace of the stress tensor for (a) P1 (b) P2 (c) P2/P1/P1

Some references for hyperelasticity

Some references on primal formulations without volumetric-locking

- discontinuous Galerkin (dG)
  - [Noels, Radovitzsky 06]
  - [ten Eyck, Lew 06]
- Hybridizable Discontinuous Galerkin (HDG)
  - [Nguyen, Peraire 12]
  - [Kabaria, Lew, Cockburn 15]
- Virtual Element Method (VEM)
  - [Chi, Beirão da Veiga, Paulino 17]
  - [Wriggers, Reddy, Rust, Hudobivnik 17]

Key ideas of Hybrid High-Order (HHO) methods

- Primal formulation with cells and faces unknowns

- Local reconstruction and stabilization
  - Gradient tensor field reconstructed in $\mathbb{P}_d^k(T;\mathbb{R}^{d\times d})$
  - Stabilization connecting cell and faces unknowns

- References
  - diffusion problem [Di Pietro, Ern, Lemaire, CMAM 14]
  - quasi-incompressible linear elasticity [Di Pietro, Ern, CMAME 15]
  - nonlinear elasticity with small def. [Botti, Di Pietro, Sochala, SINUM 17]

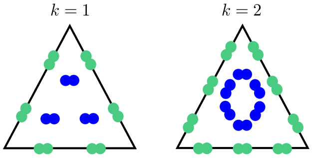


Figure 2 – Face (green) and Cell (blue) unknowns

Features of HHO methods

- Support of polytopal meshes (with possibly nonconforming interfaces)
- Arbitrary approximation order $k \geq 1$
  - h^{k+1} convergence in energy-norm
  - h^{k+2} convergence in L^2-norm with elliptic regularity
- Dimension-independent construction
- Attractive computational costs
  - Compact stencil (only neighbourhood faces)
  - Cell unknowns are eliminated locally by static condensation
  - Reduced size $N_{dofs}^{hho} \approx k^2 \mathrm{card}(\mathcal{F}^h)$ vs. $N_{dofs}^{dG} \approx k^3 \mathrm{card}(\mathcal{T}^h)$
- Local principle of virtual work (equilibrated tractions)
- HHO methods are bridged to HDG and ncVEM
  - [Cockburn, Di Pietro, Ern 16]

Hyperelasticity problem

- Let $\Omega_0 \in \mathbb{R}^d$ (d=2,3), be a bounded connected polytopal domain
- Let $\underline{\boldsymbol{f}}$ and $\underline{\boldsymbol{t}}$ be given volumetric and surface (on Γ_n) loads
- Let $\underline{\boldsymbol{u}}_D$ be a given imposed displacement on Γ_d
- We define the energy functional $\mathcal{E}$ in the reference configuration for all $\underline{\boldsymbol{v}} \in V := \left\{ \underline{\boldsymbol{v}} \in H^1(\Omega_0, \mathbb{R}^d) \,|\, \underline{\boldsymbol{v}} = \underline{\boldsymbol{u}}_D \text{ on } \Gamma_d \right\}$

$$\mathcal{E}(\underline{\boldsymbol{v}}) := \int_{\Omega_0} \Psi(\underline{\underline{\boldsymbol{F}}}(\underline{\boldsymbol{v}})) - \int_{\Omega_0} \underline{\boldsymbol{f}} . \underline{\boldsymbol{v}} \, d\Omega_0 - \int_{\Gamma_n} \underline{\boldsymbol{t}} . \underline{\boldsymbol{v}} \, d\Gamma.$$

with $\underline{\underline{\boldsymbol{F}}} := \underline{\underline{\boldsymbol{\nabla}}}_X \underline{\boldsymbol{u}} + \underline{\underline{\boldsymbol{I}}}_d$ and a strain energy density $\Psi : \mathbb{R}_+^{d \times d} \to \mathbb{R}$

- Example of Neohookean strain energy density

$$\Psi(\underline{\underline{\boldsymbol{F}}}) = \frac{\mu}{2} \left(\underline{\underline{\boldsymbol{F}}} : \underline{\underline{\boldsymbol{F}}} - d \right) - \mu \ln(\det \underline{\underline{\boldsymbol{F}}}) + \frac{\lambda}{2} (\ln(\det \underline{\underline{\boldsymbol{F}}}))^2,$$

with $\mu > 0$, $\lambda > 0$ (material constants)

Weak problem

- We assume that Ψ is polyconvex, i.e. existence of local minimizers
- Static equilibrium : stationary point(s) $\underline{\boldsymbol{u}}$ of the energy $\mathcal{E}$

$$D\mathcal{E}(\underline{\boldsymbol{u}})[\delta\underline{\boldsymbol{v}}] = 0, \ \forall \delta\underline{\boldsymbol{v}} \in H_0^1(\Omega_0, \mathbb{R}^d)$$

- Weak problem : Find $\underline{\boldsymbol{u}} \in V$ such that for all $\delta\underline{\boldsymbol{v}} \in H_0^1(\Omega_0, \mathbb{R}^d)$

$$\int_{\Omega_0} \underline{\underline{\boldsymbol{P}}}(\underline{\underline{\boldsymbol{F}}}(\underline{\boldsymbol{u}})) : \underline{\underline{\boldsymbol{\nabla}}}_X(\delta\underline{\boldsymbol{v}}) \, d\Omega_0 = \int_{\Omega_0} \underline{\boldsymbol{f}} \cdot \delta\underline{\boldsymbol{v}} \, d\Omega_0 + \int_{\Gamma_n} \underline{\boldsymbol{t}} \cdot \delta\underline{\boldsymbol{v}} \, d\Gamma.$$

with $\underline{\underline{\boldsymbol{P}}} = \partial_{\underline{\underline{\boldsymbol{F}}}}\Psi$ the first Piola–Kirchhoff stress tensor

Local DOFs space

- Let $M^h := (\mathcal{T}^h, \mathcal{F}^h)$ be a mesh of Ω_0 with $\mathcal{T}^h$ the set of cells and $\mathcal{F}^h$ the set of faces
- Let a polynomial degree $k \geq 1$, for all $T \in \mathcal{T}^h$

$$(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}) \in \underbrace{\underline{\boldsymbol{U}}_T^k}_{\text{local HHO dofs}} := \underbrace{\mathbb{P}_d^k(T; \mathbb{R}^d)}_{\text{local cell dofs}} \times \underbrace{\mathbb{P}_{d-1}^k(\mathcal{F}_{\partial T}; \mathbb{R}^d)}_{\text{local faces dofs}}.$$

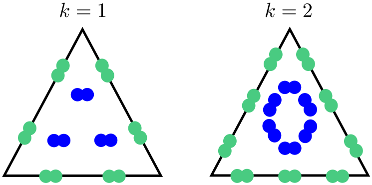


Figure 3 – Local DOFs for $k = 1, 2$. Cell unknowns eliminated by static condensation

Gradient reconstruction

$$\underline{\underline{\boldsymbol{G}}}_T^k : \underbrace{\underline{\boldsymbol{U}}_T^k}_{\text{local HHO space}} \to \underbrace{\mathbb{P}_d^k(T;\mathbb{R}^{d\times d})}_{\text{local gradient space}}$$

- The reconstructed gradient $\underline{\underline{\boldsymbol{G}}}_T^k(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T})$ solves, $\forall \underline{\underline{\boldsymbol{\tau}}} \in \mathbb{P}_d^k(T;\mathbb{R}^{d\times d})$

$$(\underline{\underline{\boldsymbol{G}}}_T^k(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}), \underline{\underline{\boldsymbol{\tau}}})_{\underline{\underline{\boldsymbol{L}}}^2(T)} = (\underline{\underline{\boldsymbol{\nabla}}}_X \underline{\boldsymbol{v}}_T, \underline{\underline{\boldsymbol{\tau}}})_{\underline{\underline{\boldsymbol{L}}}^2(T)} + (\underline{\boldsymbol{v}}_{\partial T} - \underline{\boldsymbol{v}}_T, \underline{\underline{\boldsymbol{\tau}}}\,\boldsymbol{n}_T)_{\underline{\boldsymbol{L}}^2(\partial T)}.$$

  - local scalar mass-matrix of size $\binom{k+d}{k}$ to invert (ex : $k = 2$, $d = 3$, size $= 10$)

- We define $\underline{\underline{\boldsymbol{F}}}_T^k(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}) := \underline{\underline{\boldsymbol{G}}}_T^k(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}) + \underline{\underline{\boldsymbol{I}}}_d \in \mathbb{P}_d^k(T;\mathbb{R}^{d\times d})$

- Local discrete counterpart $\mathcal{E}_T^{mech} : \underline{\boldsymbol{U}}_T^k \to \mathbb{R}$ of the energy $\mathcal{E}$

$$\mathcal{E}_T^{mech}(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}) = \int_T \left\{ \Psi(\underline{\underline{\boldsymbol{F}}}_T^k(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T})) - \underline{\boldsymbol{f}} \cdot \underline{\boldsymbol{v}}_T \right\} dT - \int_{\partial T \cap \mathcal{F}_{b,n}^h} \underline{\boldsymbol{t}} \cdot \underline{\boldsymbol{v}}_{\partial T} \, d\partial T$$

Stabilization operator

- Problem : $\underline{\underline{\boldsymbol{G}}}_T^k(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}) = \underline{\underline{\boldsymbol{0}}} \not\Rightarrow \underline{\boldsymbol{v}}_T = \underline{\boldsymbol{v}}_{\partial T} = \mathrm{cste}$
  $\Rightarrow$ We have to add a stabilization term

- Hence, we penalize the difference between the faces unknowns and the trace of the cell unknowns : $\underline{\boldsymbol{\theta}} := \underline{\boldsymbol{v}}_{\partial T} - \underline{\boldsymbol{v}}_{T|\partial T} \in \mathbb{P}_{d-1}^k(\mathcal{F}_{\partial T}; \mathbb{R}^d)$,

$$\underline{\boldsymbol{S}}_{\partial T}^k(\underline{\boldsymbol{\theta}}) = \underline{\boldsymbol{\Pi}}_{\partial T}^k(\underline{\boldsymbol{\theta}} - (\underline{\boldsymbol{I}}_d - \underline{\boldsymbol{\Pi}}_T^k)\underline{\boldsymbol{D}}_T^{k+1}(\underline{\boldsymbol{0}}, \underline{\boldsymbol{\theta}}))$$

  where $\underline{\boldsymbol{\Pi}}_{\partial T}^k$ is the L^2-projector on ∂T, $\underline{\boldsymbol{\Pi}}_T^k$ the L^2-projector on T, and $\underline{\boldsymbol{D}}_T^{k+1}$ is a reconstructed displacement field

- We define the local stabilization energy $\mathcal{E}_T^{stab} : \underline{\boldsymbol{U}}_T^k \to \mathbb{R}$

$$\mathcal{E}_T^{stab}(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}) = \frac{h_T^{-1}}{2} \|\underline{\boldsymbol{S}}_{\partial T}^k(\underline{\boldsymbol{v}}_{\partial T} - \underline{\boldsymbol{v}}_{T|\partial T})\|^2_{\underline{\boldsymbol{L}}^2(\partial T)}$$

Global DOFs space and discrete energy

- We define the global space by patching the interface DOFs

$$(\underline{\boldsymbol{v}}_{\mathcal{T}^h}, \underline{\boldsymbol{v}}_{\mathcal{F}^h}) \in \underbrace{\underline{\boldsymbol{U}}_h^k}_{\text{global HHO dofs}} := \underbrace{\left\{ \bigtimes_{T \in \mathcal{T}^h} \mathbb{P}_d^k(T; \mathbb{R}^d) \right\}}_{\text{global cells dofs}} \times \underbrace{\left\{ \bigtimes_{F \in \mathcal{F}^h} \mathbb{P}_{d-1}^k(F; \mathbb{R}^d) \right\}}_{\text{global faces dofs}}$$

and its subspaces $\underline{\boldsymbol{U}}_{h,d}^k$ and $\underline{\boldsymbol{U}}_{h,0}^k$ by imposing strongly the BC on Γ_d.

- We define the global discrete energy $\mathcal{E}_h : \underline{\boldsymbol{U}}_h^k \to \mathbb{R}$

$$\underbrace{\mathcal{E}_h(\underline{\boldsymbol{v}}_{\mathcal{T}^h}, \underline{\boldsymbol{v}}_{\mathcal{F}^h})}_{\text{global discrete energy}} = \underbrace{\sum_{T \in \mathcal{T}^h} \mathcal{E}_T^{mech}(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T})}_{\text{global mech. discrete energy}} + \underbrace{\sum_{T \in \mathcal{T}^h} \beta \, \mathcal{E}_T^{stab}(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T})}_{\text{global stabilzation energy}}$$

with β an user-dependent stabilization parameter (can be hard to choose)

Discrete problem

- We search the stationary point(s) of $\mathcal{E}_h$

$$D\mathcal{E}_h(\underline{\boldsymbol{u}}_{\mathcal{T}^h}, \underline{\boldsymbol{u}}_{\mathcal{F}^h}))[(\delta\underline{\boldsymbol{v}}_{\mathcal{T}^h}, \delta\underline{\boldsymbol{v}}_{\mathcal{F}^h})] = 0, \ \forall(\delta\underline{\boldsymbol{v}}_{\mathcal{T}^h}, \delta\underline{\boldsymbol{v}}_{\mathcal{F}^h}) \in \underline{\boldsymbol{U}}_{h,0}^k$$

- Find $(\underline{\boldsymbol{u}}_{\mathcal{T}^h}, \underline{\boldsymbol{u}}_{\mathcal{F}^h}) \in \underline{\boldsymbol{U}}_{h,d}^k$ such that

$$\begin{aligned}
&\sum_{T\in\mathcal{T}^h} (\underline{\underline{\boldsymbol{P}}}(\underline{\underline{\boldsymbol{F}}}_T^k(\underline{\boldsymbol{u}}_T, \underline{\boldsymbol{u}}_{\partial T})), \underline{\underline{\boldsymbol{G}}}_T^k(\delta\underline{\boldsymbol{v}}_T, \delta\underline{\boldsymbol{v}}_{\partial T}))_{\underline{\underline{\boldsymbol{L}}}^2(T)} \\
&+ \sum_{T\in\mathcal{T}^h} \beta h_T^{-1}(\underline{\boldsymbol{S}}_{\partial T}^k(\underline{\boldsymbol{u}}_{\partial T} - \underline{\boldsymbol{u}}_{T|\partial T}), \underline{\boldsymbol{S}}_{\partial T}^k(\delta\underline{\boldsymbol{v}}_{\partial T} - \delta\underline{\boldsymbol{v}}_{T|\partial T}))_{\underline{\boldsymbol{L}}^2(\partial T)} \\
&= \sum_{T\in\mathcal{T}^h} (\underline{\boldsymbol{f}}, \delta\underline{\boldsymbol{v}}_T)_{\underline{\boldsymbol{L}}^2(T)} + \sum_{F\in\mathcal{F}_{b,n}^h} (\underline{\boldsymbol{t}}, \delta\underline{\boldsymbol{v}}_F)_{\underline{\boldsymbol{L}}^2(F)}, \quad \forall(\delta\underline{\boldsymbol{v}}_{\mathcal{T}^h}, \delta\underline{\boldsymbol{v}}_{\mathcal{F}^h}) \in \underline{\boldsymbol{U}}_{h,0}^k
\end{aligned}$$

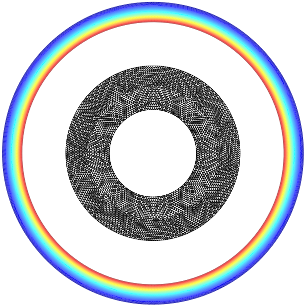
Numerical examples

- Nonlinear problem to solve (geometric and material nonlinearites)
- Iterative resolution with a Newton method
- Static condensation performed at each Newton's iteration
- Offline computations (gradient and stabilization operators precomputed)
- Implementation in the open-source library *disk++*

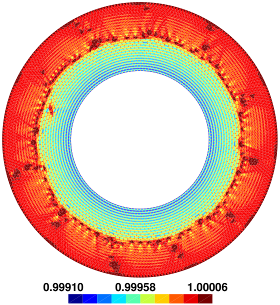
- Verification on analytical solutions :
  - Expected convergence rates (h^{k+1} in energy-norm and h^{k+2} in L^2-norm)
  - Absence of volumetric-locking in the quasi-incompressible regime
- Tested on more challenging 3D test cases (see [Kabaria, Lew, Cockburn 15])

Quasi-incompressible annulus I

- Analytical solution
- Imposed radial displacement on the inner circumference



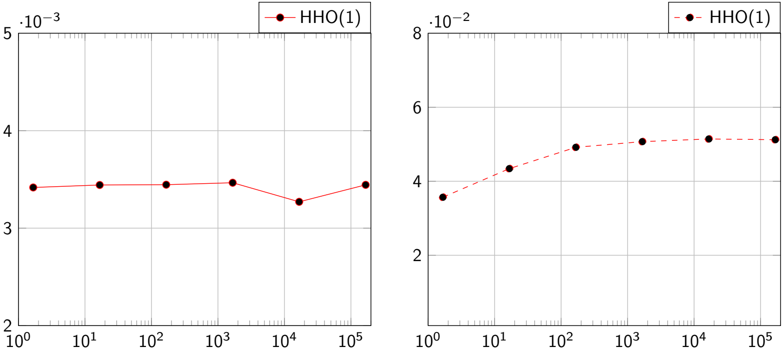
(a) Reference and final configuration



(b) Discrete Jacobian

Figure 4 – Solution for $k = 1$ and $\nu = 0.4999$

Quasi-incompressible annulus II



(a) Displacement error (b) Gradient error

Figure 5 – errors vs. λ on a fixed mesh

- The errors do not depend on λ
  $\Rightarrow$ HHO methods are robust in the incompressible limit

Sheared and compressed cylinder ($\nu = 0.45$)

- The bottom face is clamped
- Imposed vertical and horizontal displacement on the top face

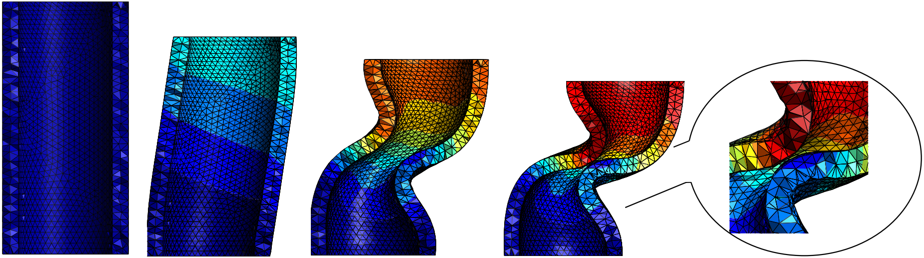


Figure 6 – Snapshots of the displacement at 0%, 40%, 80% and 100% of loading

Sphere with cavitating voids

- Growth of internal cavities under large tensile stresses
- Conforming FEM are not really robust
- Imposed radial displacement on the outer surface of the sphere
- We stop when the Newton's method fails to converge

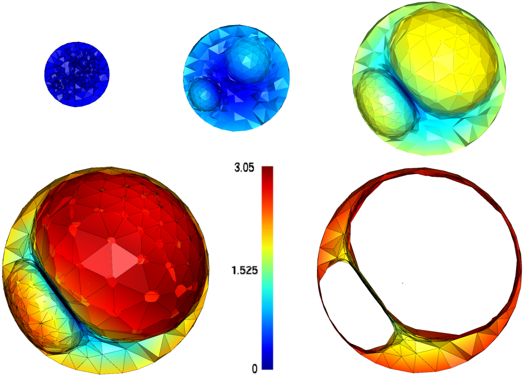


Figure 7 – Displacement for $k = 2$ at the different steps (around 250% of deformations)

Unstabilized HHO method on simplicial meshes 1

- Many trials are often necessary to find the stabilization parameter β :
  - No general theory on the choice of β
  - If β is too small $\Rightarrow$ Difficulties to converge
  - If β is too large $\Rightarrow$ The system is ill-conditioned

- The reconstructed gradient $\underline{\underline{\boldsymbol{G}}}_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}) \in \underline{\underline{\boldsymbol{R}}}$ solves, $\forall \underline{\underline{\boldsymbol{\tau}}} \in \underline{\underline{\boldsymbol{R}}}$

$$(\underline{\underline{\boldsymbol{G}}}_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}), \underline{\underline{\boldsymbol{\tau}}})_{\underline{\underline{\boldsymbol{L}}}^2(T)} = (\underline{\underline{\boldsymbol{\nabla}}}_X \underline{\boldsymbol{v}}_T, \underline{\underline{\boldsymbol{\tau}}})_{\underline{\underline{\boldsymbol{L}}}^2(T)} + (\underline{\boldsymbol{v}}_{\partial T} - \underline{\boldsymbol{v}}_T, \underline{\underline{\boldsymbol{\tau}}}\, \underline{\boldsymbol{n}}_T)_{\underline{\boldsymbol{L}}^2(\partial T)}.$$

- Sufficient conditions to reconstruct a stable gradient $\underline{\underline{\boldsymbol{G}}}_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}) \in \underline{\underline{\boldsymbol{R}}}$
  1. $\left\{\underline{\underline{\boldsymbol{\tau}}} \in \underline{\underline{\boldsymbol{R}}} \,:\, \underline{\underline{\boldsymbol{\tau}}} = \underline{\underline{\boldsymbol{\nabla}}}_X \underline{\boldsymbol{v}}_T \text{ and } \underline{\underline{\boldsymbol{\tau}}}\, \underline{\boldsymbol{n}}_T = \underline{\boldsymbol{0}}\right\} \supseteq \mathbb{P}_d^{k-1}(T; \mathbb{R}^{d\times d})$
  2. $\left\{\underline{\underline{\boldsymbol{\tau}}} \in \underline{\underline{\boldsymbol{R}}} \,:\, \underline{\underline{\boldsymbol{\tau}}} = \underline{\underline{\boldsymbol{0}}} \text{ and } \underline{\underline{\boldsymbol{\tau}}}\, \underline{\boldsymbol{n}}_T = \underline{\boldsymbol{v}}_{\partial T} - \underline{\boldsymbol{v}}_T\right\} \supseteq \mathbb{P}_d^{k}(T; \mathbb{R}^{d\times d})$

$\Rightarrow$ Control independently the volumetric and normal components of $\underline{\underline{\boldsymbol{\tau}}} \in R$

- For approximation results $\mathbb{P}_d^k(T; \mathbb{R}^{d\times d}) \subseteq \underline{\underline{\boldsymbol{R}}}$

Unstabilized HHO method on simplicial meshes 2

- Original idea for dG : [John, Neilan, Smears 16]
  - Based on the properties of the Raviart–Thomas space
- Gradient reconstruction in $\underline{\underline{\boldsymbol{R}}} = \mathbb{P}_d^{k+1}(T; \mathbb{R}^{d\times d})$ (larger space)
  - ex : $k = 2$, $d = 3$, size $= 20$ for $\mathbb{P}_d^{k+1}(T; \mathbb{R}^{d\times d})$ vs 10 for $\mathbb{P}_d^{k}(T; v^{d\times d})$
- No additional stabilization is needed
- Lower convergence rates (h^k in energy-norm and h^{k+1} in L^2-norm)
- Comparable numerical cost vs. stabilized HHO (sHHO) methods
- Better results for the cavitation problem ($r_{max} = 2.52$ vs. $r_{max}^{sHHO} = 2.13$)
- Other choice : $\underline{\underline{\boldsymbol{R}}} = \mathbb{RT}^k(T; \mathbb{R}^{d\times d})$
  - Smaller than $\mathbb{P}_d^{k+1}(T; \mathbb{R}^{d\times d}) \supset \mathbb{RT}^k(T; \mathbb{R}^{d\times d})$
  - Same convergence than sHHO
  - Less robust for very large deformations

Conclusions and perspectives

- Conclusion :
  - Adaptation of HHO methods to hyperelastic material with finite deformations
  - Absence of volumetric-locking
  - Variant of HHO method without stabilization
- Perspectives of this work :
  - Extension to finite plasticity
  - Introduction of contact and friction
  - Implementation in `code_aster` (in progress)

code_aster

Thank you for your attention

email : nicolas.pignet@enpc.fr
code : https ://github.com/datafl4sh/diskpp

Reference : M. Abbas, A. Ern and NP, "Hybrid High-Order methods for finite deformations of hyperelastic materials", Comput. Mech. (2018)