Colloquium du CERMICS

Reduced order model approach for imaging with waves

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Reduced order model approach for imaging with waves

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Motivation: Sensor array imaging

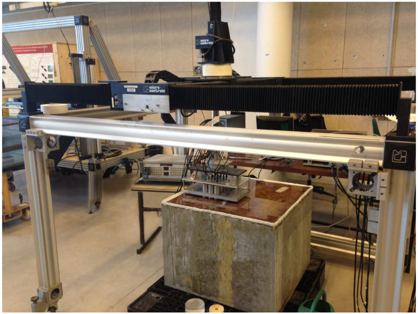
- Sensor array imaging (echography in medical imaging, sonar, non-destructive testing, seismic exploration, etc) has two steps:
  - data acquisition: an unknown medium is probed with waves; waves are emitted by a source (or a source array) and recorded by a receiver (or a receiver array).
  - data processing: the recorded signals are processed to identify the quantities of interest (reflector locations, etc).

Example:
Ultrasound echography

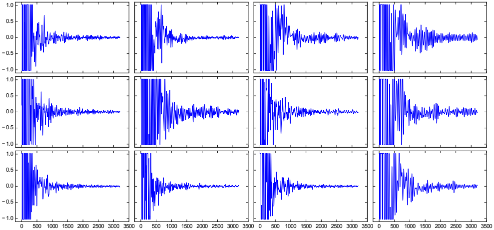
$\longrightarrow$

- Mathematically: Ill-posed inverse problems.

Example: Ultrasound in concrete

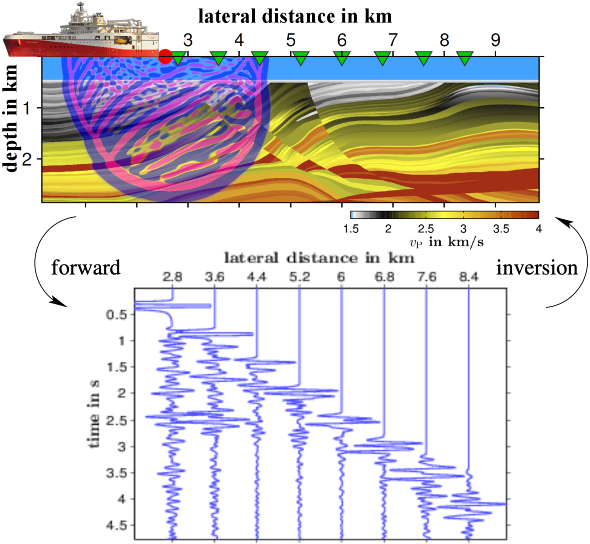


Experience: nondestructive testing



Data: recorded signals

Example: Reflection seismology



Velocity estimation problem

- *Direct problem:* Given the velocity map $c = (c(x))_{x \in \Omega}$ compute the wavefield solution of the wave equation

$$[\partial_t^2 - c^2(x)\Delta]p^{(s)}(t,x) = f(t)\delta(x - x_s), \qquad t \in \mathbb{R}, \quad x \in \Omega \subset \mathbb{R}^d,$$

starting from $p^{(s)}(t,x) = 0$, $t \ll 0$, + boundary conditions at $\partial\Omega$.
At the locations of the receivers:

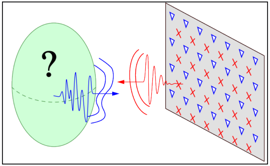
$$d_{r,s}(t) = p^{(s)}(t, x_r), \quad r,s = 1,..,N$$

$\hookrightarrow$ forward map

$$\mathcal{D} : c \mapsto \mathbf{d}$$

where $\mathbf{d} = ((d_{r,s}(t))_{r,s=1}^N)_{t \in [t_{\min}, t_{\max}]}$, is the array response matrix.

- *Inverse problem:*
Given the time-resolved measurements $\mathbf{d}$,
determine the velocity map c.



Full Waveform Inversion (FWI)

- FWI fits data with the model prediction (least-square minimization):

$$\hat{c} = \underset{c}{\operatorname{argmin}}\ \mathcal{O}_{FWI}[c],$$

$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}_{meas}\|^2 = \sum_{r,s=1}^{N} \int_{t_{\min}}^{t_{\max}} |\mathcal{D}[c](t)_{r,s} - d_{meas}(t)_{r,s}|^2 dt$$

  Cf [Virieux and Operto 2009].
- Problem: The objective function $\mathcal{O}_{FWI}[c]$ is not convex in c.
  $\hookrightarrow$ optimization needs hard to get good initial guess.

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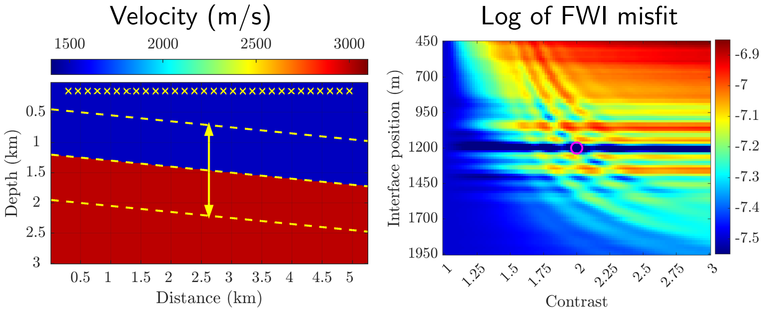
Cf [Virieux and Operto 2009].
- Problem: The objective function $\mathcal{O}_{FWI}[c]$ is not convex in c.
$\hookrightarrow$ optimization needs hard to get good initial guess.

- Regularization: $\hat{c} = \mathrm{argmin}_c \{\mathcal{O}_{FWI}[c] + \lambda \mathrm{Reg}[c]\}$, with $\mathrm{Reg}[c] = \|c\|_{L^2}^2, \|c\|_{L^1}, \|c\|_{\mathrm{TV}}$, ... (Bayesian interpretation).
- Layer stripping: Proceed hierarchically from the shallow part to the deep part [Wang et al. 2009]
- Frequency hopping: Successive inversion of subdata sets of increasing high-frequency content [Bunks et al. 1995]
- Optimal transport: Wasserstein distance instead of least-squares [Engquist et al., 2014, Métivier et al. 2016]

Topography of the FWI objective function



- Search velocity has two parameters: the bottom velocity and depth of the interface (the angle and top velocity are known).
- Probing pulse is a modulated Gaussian pulse with central frequency $6Hz$ and bandwidth $4Hz$ ($\lambda \simeq 300m$ at $10Hz$).

  $N = 30$ sensors; $N_{\rm t} = 39$ time samples at interval $\tau = 0.0435$s.
- Objective function:

$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}_{meas}\|^2$$

$\hookrightarrow$ Many local minima.

Objective

- Objective: Find a convex formulation of FWI.
- Proposed approach: find a (nonlinear) mapping $\mathcal{R}(\mathbf{d})$ such that:

$$\mathcal{O}[c] = \|\mathcal{R}(\mathcal{D}[c]) - \mathcal{R}(\mathbf{d}_{meas})\|^2$$

has better convexity properties than

$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}_{meas}\|^2$$

Remark : We can think of the mapping $\mathcal{R}$ as a nonlinear preconditioner of the forward mapping $\mathcal{D}$.

Towards the ROM objective function

- *Ideal objective function 1:*

$$\mathcal{O}[c] = \|c - c_{meas}\|^2 = \int_\Omega |c(x) - c_{meas}(x)|^2 dx$$

but c_{meas} is not observed (i.e., cannot be extracted from $\mathbf{d}_{meas}$) !

Towards the ROM objective function

- Let us consider the wave operator

$$\mathcal{A}[c] = -c(x)\Delta\big[c(x)\,\cdot\,\big]$$

- Galerkin method to approximate the operator $\mathcal{A}$ by a matrix:

- consider a space of (piecewise polynomial) functions with basis $(\Psi_l(x))_{l=1}^L$,
- consider the row vector field $\boldsymbol{\Psi}(x) = (\Psi_1(x), \dots, \Psi_L(x))$ and define:

$$\mathbf{A}^\Psi = \int_\Omega dx\, \boldsymbol{\Psi}(x)^T \mathcal{A} \boldsymbol{\Psi}(x) \in \mathbb{R}^{L\times L}$$

- *Ideal objective function 2:*

$$\mathcal{O}[c] = \|\mathbf{A}^\Psi[c] - \mathbf{A}^\Psi_{meas}\|^2 = \sum_{l,l'=1}^L \big|A^\Psi[c]_{ll'} - A^\Psi_{meas,ll'}\big|^2$$

but $\mathbf{A}^\Psi_{meas}$ is not observed !

The ROM matrix

- Our Galerkin approximation space:

- consider a time discretization $\{t_j = j\tau\}_{0\leq j<N_{\rm t}}$ with uniform stepping τ,
- gather the waves $p^{(s)}(t,x)$ evaluated at $t=t_j$ for all the N sources:

$$\boldsymbol{p}_j(x) = \Big(p^{(1)}(t_j,x),\dots,p^{(N)}(t_j,x)\Big)\,, \quad x\in\Omega.$$

(note: apply first a linear preprocessing).
- organize the first $N_{\rm t}$ snapshots in the $NN_{\rm t}$ dimensional row vector field:

$$\boldsymbol{U}(x) = (\boldsymbol{p}_0(x),\dots,\boldsymbol{p}_{N_{\rm t}-1}(x))\,, \quad x\in\Omega.$$

- apply Gram-Schmidt orthogonalization onto $\boldsymbol{U}(x) = \boldsymbol{V}(x)\mathbf{R}$.

- Define ROM matrix:

$$\mathbf{A}^{rom} = \int_\Omega dx\, \boldsymbol{V}(x)^T \mathcal{A}\boldsymbol{V}(x) \in \mathbb{R}^{NN_{\rm t}\times NN_{\rm t}}.$$

- *Ideal objective function 3:*

$$\mathcal{O}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom}_{meas}\|^2$$

but $\mathbf{A}^{rom}_{meas}$ is not observed (neither $\mathcal{A}$ nor $\boldsymbol{V}(x)$ is observed) !

The ROM matrix

- Our Galerkin approximation space:

- consider a time discretization $\{t_j = j\tau\}_{0\leq j<N_\mathrm{t}}$ with uniform stepping τ,
- gather the waves $p^{(s)}(t,x)$ evaluated at $t = t_j$ for all the N sources:

$$\boldsymbol{p}_j(x) = \left(p^{(1)}(t_j,x),\dots,p^{(N)}(t_j,x)\right), \quad x \in \Omega.$$

(note: apply first a linear preprocessing).

- organize the first N_t snapshots in the NN_t dimensional row vector field:

$$\boldsymbol{U}(x) = \left(\boldsymbol{p}_0(x),\dots,\boldsymbol{p}_{N_\mathrm{t}-1}(x)\right), \quad x \in \Omega.$$

- apply Gram-Schmidt orthogonalization onto $\boldsymbol{U}(x) = \boldsymbol{V}(x)\mathbf{R}$.

- Define ROM matrix:

$$\mathbf{A}^{rom} = \int_\Omega dx\, \boldsymbol{V}(x)^T \mathcal{A} \boldsymbol{V}(x) \in \mathbb{R}^{NN_\mathrm{t}\times NN_\mathrm{t}}.$$

- **Proposition**: *The ROM matrix $\mathbf{A}^{rom}$ can be extracted from the measurements $\mathbf{d}$, without knowing $\mathcal{A}$ nor $\boldsymbol{V}(x)$.*

$\hookrightarrow$ $\mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom}_{meas}\|^2$ *is a legitimate objective function.*

Proof of the proposition

First step: *Linear preprocessing*.

- Define the new data matrix $\mathbf{d}^f(t)$:

$$\mathbf{d}^f(t) = [-f'(-\cdot) *_t \mathbf{d}](t) + [-f'(-\cdot) *_t \mathbf{d}](-t).$$

Second step: *Expression of the new data entries as wave correlations*.

- Introduce the solution $u^{(s)}(t,x)$ of the homogeneous wave equation

$$(\partial_t^2 + \mathcal{A})u^{(s)}(t,x) = 0, \qquad t > 0, \quad x \in \Omega,$$

with boundary conditions on $\partial\Omega$, with initial state

$$u^{(s)}(0,x) = u_0^{(s)}(x) = \left|\hat{f}\left(\sqrt{\mathcal{A}}\right)\right| \delta(x - x_s), \qquad \partial_t u^{(s)}(0,x) = 0.$$

It has the form

$$u^{(s)}(t,x) = \cos\left(t\sqrt{\mathcal{A}}\right) u_0^{(s)}(x).$$

$\to$ The entries of $\mathbf{d}^f(t)$ can be expressed as wave correlations:

$$d^f_{r,s}(t) = \int_\Omega dx \, u_0^{(r)}(x) u^{(s)}(t,x).$$

Proof of the proposition

Third step: *Definition of the ROM.*
Let $\tau > 0$ be fixed.
• Gather the $N_{\rm t}$ snapshots for all the N sources in the row vector fields

$$\boldsymbol{u}_j(x) = \Big(u^{(1)}(j\tau, x), \ldots, u^{(N)}(j\tau, x) \Big), \qquad 0 \leq j < N_{\rm t}.$$

• Organize the first $N_{\rm t}$ snapshots in the $NN_{\rm t}$ dimensional row vector field:

$$\boldsymbol{U}(x) = \left(\boldsymbol{u}_0(x), \ldots, \boldsymbol{u}_{N_{\rm t}-1}(x) \right), \quad x \in \Omega.$$

• Apply Gram-Schmidt orthogonalization onto $\boldsymbol{U}(x) = \boldsymbol{V}(x)\mathbf{R}$.
(note: we have $\int_\Omega dx\, \boldsymbol{V}(x)^T \boldsymbol{V}(x) = \mathbf{I}_{NN_{\rm t}}$).
• Define

$$\mathbf{A}^{rom} = \int_\Omega dx\, \boldsymbol{V}(x)^T \mathcal{A} \boldsymbol{V}(x)$$

Proof of the proposition

Fourth step: *Expression of the ROM in terms of mass and stiffness.*

- Define the $NN_\mathrm{t} \times NN_\mathrm{t}$ "mass" and "stiffness" matrices:

$$\mathbf{M} = \int_\Omega dx\, \boldsymbol{U}^T(x)\boldsymbol{U}(x), \qquad \mathbf{S} = \int_\Omega dx\, \boldsymbol{U}^T(x)\mathcal{A}\boldsymbol{U}(x)$$

- Since $\boldsymbol{U}(x) = \boldsymbol{V}(x)\mathbf{R}$, we get

$$\begin{aligned}\mathbf{M} &= \mathbf{R}^T \int_\Omega dx\, \boldsymbol{V}^T(x)\boldsymbol{V}(x)\mathbf{R} \\ &= \mathbf{R}^T\mathbf{R}\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}^{rom} &= \int_\Omega dx\, \boldsymbol{V}(x)^T\mathcal{A}\boldsymbol{V}(x) = \mathbf{R}^{-T}\int_\Omega dx\, \boldsymbol{U}(x)^T\mathcal{A}\boldsymbol{U}(x)\mathbf{R} \\ &= \mathbf{R}^{-T}\mathbf{S}\mathbf{R}\end{aligned}$$

$\hookrightarrow$ $\mathbf{A}^{rom}$ can be expressed in terms of $\mathbf{M}$ and $\mathbf{S}$.

Proof of the proposition

Fifth step: *Expression of the ROM in terms of data*.
The $N \times N$ blocks of the mass matrix $\mathbf{M}$ are

$$\begin{aligned}
\mathbf{M}_{i,j} &= \langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle_{L^2(\Omega)} = \langle \cos\big(i\tau\sqrt{\mathcal{A}}\big)\boldsymbol{u}_0, \cos\big(j\tau\sqrt{\mathcal{A}}\big)\boldsymbol{u}_0 \rangle_{L^2(\Omega)} \\
&= \langle \boldsymbol{u}_0, \cos\big(i\tau\sqrt{\mathcal{A}}\big)\cos\big(j\tau\sqrt{\mathcal{A}}\big)\boldsymbol{u}_0 \rangle_{L^2(\Omega)} \\
&= \frac{1}{2}\langle \boldsymbol{u}_0, \big[\cos\big((i+j)\tau\sqrt{\mathcal{A}}\big) + \cos\big(|i-j|\tau\sqrt{\mathcal{A}}\big)\big]\boldsymbol{u}_0 \rangle_{L^2(\Omega)} \\
&= \frac{1}{2}\langle \boldsymbol{u}_0, \boldsymbol{u}_{i+j} + \boldsymbol{u}_{|i-j|} \rangle_{L^2(\Omega)} \\
&= \frac{1}{2}\Big(\mathbf{d}^f((i+j)\tau) + \mathbf{d}^f(|i-j|\tau)\Big), \quad 0 \le i,j < N_{\mathrm{t}}.
\end{aligned}$$

Idem for the stiffness matrix $\mathbf{S}$.
$\hookrightarrow$ $\mathbf{M}$ and $\mathbf{S}$ can be expressed in terms of the data matrix $\mathbf{d}^f$.

Algorithm for data-driven ROM matrix

Input: The matrices $\mathbf{d}(t) = (d_{r,s}(t))_{r,s=1}^{N}$ of measurements.

1. Compute $d^f_{r,s}(t) = [-f'(-\cdot) *_t d_{r,s}](t) + [-f'(-\cdot) *_t d_{r,s}](-t)$ and

$$\mathbf{D}_j = \mathbf{d}^f(j\tau), \quad 0 \leq j \leq 2N_{\rm t} - 2.$$

2. Compute $\ddot{\mathbf{D}}_j = \ddot{\mathbf{d}}^f(j\tau)$, for $j = 0, \ldots, 2N_{\rm t} - 2$ with $\ddot{d}^f_{r,s}(t) = \partial_t^2 d^f_{r,s}(t)$ using, e.g., the Fourier transform.

3. Calculate $\mathbf{M}, \mathbf{S} \in \mathbb{R}^{NN_{\rm t} \times NN_{\rm t}}$ with the block entries

$$\mathbf{M}_{i,j} = \frac{1}{2}(\mathbf{D}_{i+j} + \mathbf{D}_{|i-j|}) \in \mathbb{R}^{N \times N},$$
$$\mathbf{S}_{i,j} = -\frac{1}{2}(\ddot{\mathbf{D}}_{i+j} + \ddot{\mathbf{D}}_{|i-j|}) \in \mathbb{R}^{N \times N},$$

for $0 \leq i, j \leq N_{\rm t} - 1$.

4. Perform block Cholesky factorization $\mathbf{M} = \mathbf{R}^T \mathbf{R}$.

Output: $\mathbf{A}^{rom} = \mathbf{R}^{-T} \mathbf{S} \mathbf{R}^{-1}$.

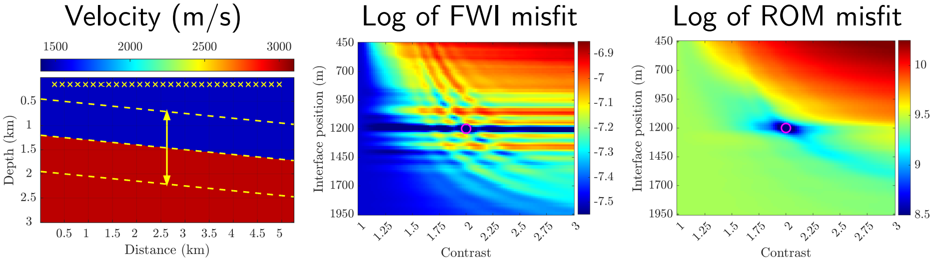
ROM objective function

- ROM misfit function:

$$\mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom}_{meas}\|^2$$

  where $\mathbf{A}^{rom}[c]$ is computed from $\mathcal{D}[c]$ and $\mathbf{A}^{rom}_{meas}$ is computed from $\mathbf{d}_{meas}$.
- For a rich enough space of snapshots, the ROM matrix $\mathbf{A}^{rom}$ contains roughly the same information as $\mathcal{A} = -c(x)\Delta\big[c(x)\cdot\big]$.
  $\hookrightarrow$ The ROM misfit function should have nice convexity properties.
- Conjecture: "rich enough" would mean for sensors all around the domain of interest, separated by roughly half a wavelength, for time sampling satisfying the Nyquist criterium.
  $\hookrightarrow$ Conjecture proved only in special situations.

Topographies of the FWI and ROM objective functions



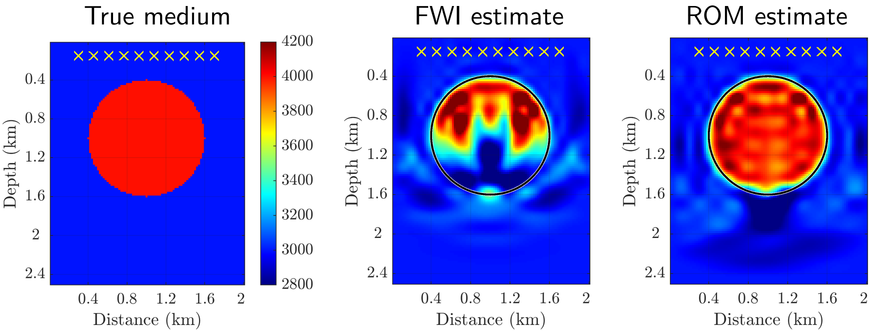
- Search velocity has two parameters: the contrast and the depth of the interface (the angle and top velocity are known).
- FWI objective function:

$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}_{meas}\|^2$$

- ROM objective function:

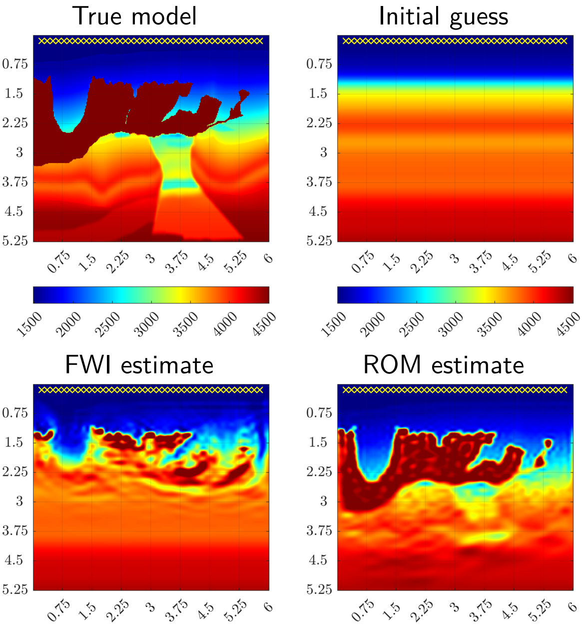
$$\mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom}_{meas}\|^2$$

Camembert model



- Probing pulse is a modulated Gaussian pulse with central frequency $6Hz$ and bandwidth $4Hz$ ($\lambda = 300m$ at $10Hz$).
- Search velocity: $c(x, \boldsymbol{\eta}) = c_o + \sum_l \eta_l \phi_l(x)$, $\boldsymbol{\eta} = (\eta_l)_{l=1}^{L}$.
- $\phi_l(x)$ are Gaussian peaks with centers on a regular grid, $L = 400$, with width 60m (0.2λ).
- FWI minimizes $\mathcal{O}_{FWI}(\boldsymbol{\eta}) = \|\mathcal{D}[c(\boldsymbol{\eta})] - \mathbf{d}_{meas}\|^2 + \mu\|\boldsymbol{\eta}\|^2$
- ROM minimizes $\mathcal{O}_{ROM}(\boldsymbol{\eta}) = \|\mathbf{A}^{rom}[c(\boldsymbol{\eta})] - \mathbf{A}^{rom}_{meas}\|^2 + \mu\|\boldsymbol{\eta}\|^2$

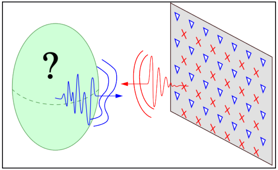
Salt body (BP - model)



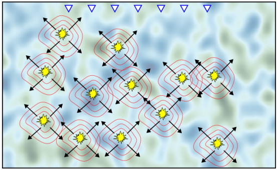
One limitation and two extensions

- One limitation of the method:
  We need co-located sources and receivers.
- Extension to passive imaging:
  Consider a *receiver* array at $(x_r)_{r=1}^N$ recording signals transmitted by noise sources (uncontrolled, opportunistic sources).
  Compute the cross correlation matrix of the recorded signals.
  $\rightarrow$ The ROM procedure is natural in the passive framework, since the cross correlation matrix gives directly the data matrix
  $\mathbf{d}^f(t) = (d^f_{r,r'}(t))_{r,r'=1}^N$.
- Extension to vector waves.

Passive imaging



Active acquisition



Passive acquisition

- Consider the solution $p(t,x)$ of the wave equation

$$\partial_t^2 p - c^2(x)\Delta p = s(t,x), \qquad t \in \mathbb{R}, \quad x \in \Omega \subset \mathbb{R}^d,$$

where $s(t,x)$ is a zero-mean, stationary in time random process with

$$\langle s(t_1,y_1)s(t_2,y_2)\rangle = F(t_1-t_2)K(y_1)\delta(y_1-y_2)$$

- The passive data set is $((p(t,x_r))_{r=1}^N)_{t\in[0,T]}$, with $T \gg 1$.
- The empirical cross correlation of the recorded waves at x_r and $x_{r'}$ is

$$C_T(\tau,x_r,x_{r'}) = \frac{1}{T}\int_0^T dt\, p(t,x_r)p(t+\tau,x_{r'})$$

Passive imaging

- The statistical cross correlation

$$C^{(1)}(\tau, x_r, x_{r'}) = \big\langle C_T(\tau, x_r, x_{r'}) \big\rangle$$

is independent of T by stationarity of the noise sources.

- The statistical stability follows from the ergodicity of the noise sources:

$$C_T(\tau, x_r, x_{r'}) \overset{T \to +\infty}{\longrightarrow} C^{(1)}(\tau, x_r, x_{r'}),$$

in probability [Garnier et al. 2016].

• **Proposition.** We have, for any $r, r' = 1, \ldots, N$,

$$\partial_\tau^2 C^{(1)}(\tau, x_r, x_{r'}) = -\frac{1}{4} d^f_{r,r'}(\tau),$$

where $\mathbf{d}^f(t)$ is the active data matrix obtained with a source signal $f(t)$ that satisfies $|\hat{f}(\omega)| = \hat{F}(\omega)^{1/2}$ ($\hat{F}$ = power spectral density of the noise sources).

• **Corollary.** *The passive data (cross correlation matrix) can be used in the ROM algorithm (no preprocessing).*

An extension to first-order systems

- General framework: The vectorial wave field $\boldsymbol{\psi}_{\boldsymbol{\varepsilon}} \in \mathbb{R}^m$ satisfies

$$\partial_t \boldsymbol{\psi}_{\boldsymbol{\varepsilon}}(t, \boldsymbol{x}) + \mathcal{L}\boldsymbol{\psi}_{\boldsymbol{\varepsilon}}(t, \boldsymbol{x}) = s(t)\boldsymbol{f}_{\boldsymbol{\varepsilon}}(\boldsymbol{x}), \qquad t \in \mathbb{R},\ \boldsymbol{x} \in \Omega \subset \mathbb{R}^d,$$

  where $\mathcal{L}$ is skew-adjoint, $\boldsymbol{f}_{\boldsymbol{\varepsilon}}(\boldsymbol{x})$ models a source localized at point $\boldsymbol{x}_{\varepsilon_1}$, with polarization indexed by ε_2 ($\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in \mathbb{N}^2$).
- Can model acoustics, elasticity, and electromagnetism.
- The array response matrix is:

$$\mathbf{d}_{\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}}(t) = \int_\Omega d\boldsymbol{x}\ [\boldsymbol{f}_{\boldsymbol{\varepsilon}'}(\boldsymbol{x})]^T\, \boldsymbol{\psi}_{\boldsymbol{\varepsilon}}(t, \boldsymbol{x}).$$

- Main hypothesis: Requires multi-dimensional, collocated sources and receivers.
- Main goal: Multiparametric inversion.

First-order acoustic system

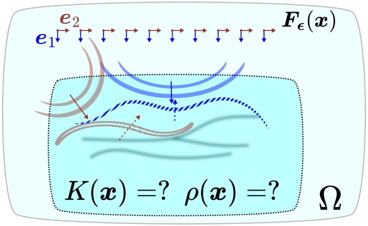
The acoustic wave equation:

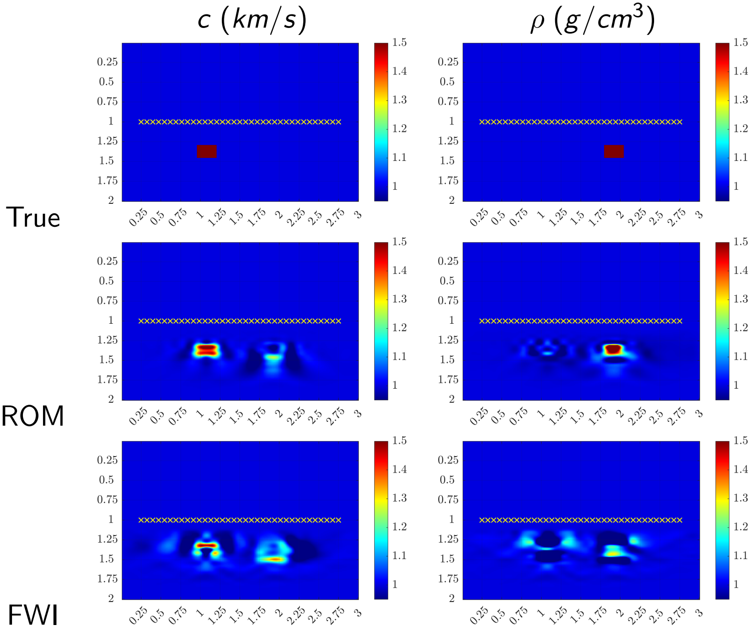
$$\partial_t \boldsymbol{u}_\varepsilon(t,\boldsymbol{x}) + \rho^{-1}(\boldsymbol{x})\nabla p_\varepsilon(t,\boldsymbol{x}) = s(t)\boldsymbol{F}_\varepsilon(\boldsymbol{x}),$$
$$\partial_t p_\varepsilon(t,\boldsymbol{x}) + K(\boldsymbol{x})\nabla\cdot\boldsymbol{u}_\varepsilon(t,\boldsymbol{x}) = 0,$$

can be formulated in the general first-order form with ($c = \sqrt{K/\rho}$):

$$\boldsymbol{\psi}_\varepsilon(t,\boldsymbol{x}) = \begin{pmatrix} \sqrt{\rho(\boldsymbol{x})}\boldsymbol{u}_\varepsilon(t,\boldsymbol{x}) \\ \frac{1}{\sqrt{K(\boldsymbol{x})}}p_\varepsilon(t,\boldsymbol{x}) \end{pmatrix}, \qquad \boldsymbol{f}_\varepsilon(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{F}_\varepsilon(\boldsymbol{x}) \\ 0 \end{pmatrix},$$

$$\mathcal{L} = \begin{pmatrix} 0 & \frac{1}{\sqrt{\rho(\boldsymbol{x})}}\mathrm{grad}\left[c(\boldsymbol{x})\sqrt{\rho(\boldsymbol{x})}\cdot\right] \\ c(\boldsymbol{x})\sqrt{\rho(\boldsymbol{x})}\mathrm{div}\left[\frac{1}{\sqrt{\rho(\boldsymbol{x})}}\cdot\right] & 0 \end{pmatrix}.$$




$c\ (km/s)$
$\rho\ (g/cm^3)$
True
ROM
FWI
0.25
0.5
0.75
1
1.25
1.5
1.75
2
2.25
2.5
2.75
3
1
1.1
1.2
1.3
1.4
1.5

Conclusions

- The ROM is an approximation of the wave operator on a space defined by the snapshots of the wavefield.
- This space is not known and neither is the wave operator.
- Yet, we can compute the ROM from the data !
- We can then formulate a velocity estimation algorithm that minimizes the ROM misfit and that avoids cycle skipping and other problems.
- The method can be applied to active and passive imaging.
- To be continued (for vector waves).

L Borcea, J Garnier, AV Mamonov, J Zimmerling, When data driven reduced order modeling meets waveform inversion, SIAM Rev. 66 (3), 2024, 501-532.

L Borcea, J Garnier, AV Mamonov, J Zimmerling, Waveform inversion with a data driven estimate of the internal wave, SIAM Journal on Imaging Sciences 16 (1), 2023, 280-312.

L Borcea, J Garnier, AV Mamonov, J Zimmerling, Waveform inversion via reduced order modeling, Geophysics 88 (2), 2023, R175-R191.